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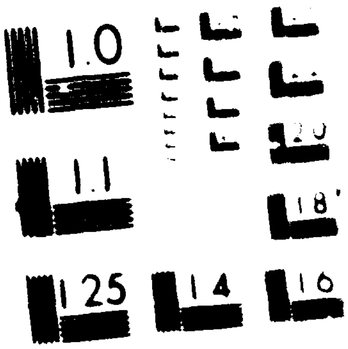
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by

Paul Dupuis and Harold J. Kushner

February 1987 LCDS/CCS #87-21

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ABSTRACT

W.p.1 convergence results are obtained for stochastic recursive approximation algorithms under very general conditions. The gain sequence $\{a_n\}$ can go to zero very slowly and state-dependent noise, discontinuous dynamical equations and the projected or constrained algorithm are all treated. The basic technique is the theory of large deviations. Prior results obtained via this theory are extended in many directions. Let $\dot{x} = \bar{b}(x)$ denote the 'mean' equation for the algorithm, let $\delta > 0$ be given, and let $G(\theta)$ be a neighborhood of a stable point θ of that ODE. Then, asymptotic upper bounds to $a_N \log P\{X_n \notin G(\theta), n \geq N \mid |X_N - \theta| \leq \delta\}$ are obtained. These are often more informative than the usual classical rate of convergence results (which use a 'local linearization') and, furthermore, are obtained for the constrained and non-smooth cases, for which there are no 'rate of convergence' results.

Key Words: Stochastic approximation, large deviations, recursive algorithms, errors for tracking systems

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1. INTRODUCTION

We obtain w.p.1 convergence results as well as useful (non-classical) estimates of 'rate of convergence' for fairly general stochastic approximation (SA) processes of the form (1.1), via the theory of large deviations (R^r = Euclidean r-space)

$$X_{n+1} = X_n + a_n b_n(X_n, \xi_n), \quad X_n \in R^r, \quad 0 < a_n \to 0, \quad \Sigma a_n = \infty. \tag{1.1}$$

We also treat the projection algorithm (1.2), where π_G denotes the nearest point of a compact convex set G.

$$X_{n+1} = \pi_G(X_n + a_n b_n(X_n, \xi_n)) \tag{1.2}$$

Such algorithms have been the subject of considerable attention [1] - [4], under a great variety of conditions. They appear in various guises in many places in control and communication theory.

In (1.1), the $\{\xi_n\}$ is a random process, which might be state dependent itself and which takes values in a compact metric space M. The b_n might simply be a function of X_n, ξ_n. More generally, we allow $\{b_n\}$ to be a sequence of vector valued (R^r) mutually independent, but not necessarily stationary *random fields* parametrized by X_n, ξ_n. In this case b_n is characterized by the distribution function (which will depend on n in the non-stationary case)

$$P(b_n \in B \mid X_i, \xi_i, b_{i-1}, i \leq n) = P(b_n \in B \mid X_n, \xi_n). \tag{1.3}$$

We suppose that $|b_n| \leq K < \infty$ for some constant K. There are many applications where the random field notation is useful since it is awkward or difficult to express explicitly all the random variables which might be involved (e.g., the X_n, ξ_n might determine other random variables which are used, in turn, to calculate X_{n+1} from X_n). For example, consider an adaptive routing problem, where X_n denotes the routing parameter and ξ_n the (vector) buffer occupancies at time n. Then b_n might be a random variable which depends on 'arrivals', 'completed services', 'acceptances of arrivals', etc. at time n, and each of these might be related to X_n, ξ_n only statistically – but the exact relation is either too complicated to write (perhaps involving a sum of indicator functions of various possible events) or not necessary to write.

If b_n is simply a function of X_n, ξ_n, $(b(X_n,\xi_n))$, then we call it a *deterministic random field*. Even in this case, the ξ_n might be state dependent, correlated, or $b(\cdot)$ might be discontinuous. If $\{b_n\}$ is a deterministic random field, we write it simply as $b(X_n,\xi_n)$. Of course, since $\{\xi_n\}$ is a random sequence, $\{b(X_n,\xi_n)\}$ is not deterministic, in the usual sense.

Perhaps the weak convergence based methods [3], [5], [6] are the most powerful general methods for dealing with the asymptotic properties of (1.1) or (1.2). The conditions for the validity of such methods are often readily verifiable. One common approach is to derive an ODE (ordinary differential equation) for the 'mean' dynamics $\dot{x} = \bar{b}(x) = Eb(x,\xi)$ (where this is well defined) and to show that the asymptotic path of $\{X_n\}$ is arbitrarily close to that of the asymptotic solutions to $\dot{x} = \bar{b}(x)$ in the sense of the weak convergence theory. Typically, under some stability property of the ODE, this method locates the points (or point) near which $\{X_n\}$ spends 'nearly all of its time'. Nevertheless, there is still considerable interest in actual w.p.1. convergence. A powerful method would use a weak convergence approach to find the 'asymptotic' points or sets, and then use a 'local' method to show w.p.1 convergence of $\{X_n\}$ to an appropriate stable point of the ODE, under the usual condition that some compact set in its domain of attraction is entered infinitely often (which would itself often be shown by a weak convergence based method).

Among methods that can be used to prove w.p.1 convergence, those based on the theory of large deviations have a number of advantages. They can handle a more general (and much more 'slowly converging') gain sequence $\{a_n\}$ then the classical methods. (They can have difficulty with problems where the qth moments of the ξ_n or $b_n(x,\xi_n)$ grow too fast as $q \to \infty$ (say, faster than those for b_n = Gaussian), but this rarely seems to be a serious problem in applications.) Due to recent advances in the theory of large deviations, we can now also treat problems with state dependent noise and discontinuous dynamics as well as constrained problems. These facts imply the availability of a rather powerful technique for getting w.p.1. convergence. The state dependent noise is more general than allowed in [3], [7]. The mathematical development here seems to more complicated than the powerful 'martingale' based methods of [4], [8]. However, we can handle more slowly (and erratically) converging gains, the constrained case, a different class of state dependent noise cases, the random field model, and get a very informative estimate of the rate of convergence even when the classical 'local' smoothness conditions are violated. This latter point is particularly important.

Typically, the large deviations estimates involve both an upper and a lower bound for a (suitably normalized) probability of a 'rare' event (say the event that the stochastic approximation (asymptotically) escapes from a small neighborhood of a stable point of $\dot{x} = \bar{b}(x)$). To get the w.p.1 convergence here, only an upper bound is needed, and this allows a result under weaker conditions than would be required if both bounds were desired. The upper bound serves as a useful indicator of the rate of convergence, perhaps even more useful than that obtained by the classical methods. It is often obtainable even for non-stationary problems, in contrast to the classical 'rate' results.

The 'rate' calculated by the classical methods is just the asymptotic variance of $(X_n - \theta)/a_n^{1/2}$, where θ is the limit point. Its derivation requires a certain 'regularity' in the way $a_n \to 0$, and a local expansion of the dynamics about θ. Assuming appropriate smoothness (usually twice differentiability of $b(x,\xi)$ at $x = \theta$, which is not needed by the large deviations method) of b for x near θ, the classical rate depends only on the gradient of $Eb(x,\xi)$ for $x = \theta$ and on the statistics of $\{b(\theta,\xi_n)\}$. In many applications, one is more interested in an (suitably normalized) estimate of the probability that the path $\{X_n, \infty > n \geq N\}$ will escape from some given neighborhood of θ for large N. This would involve the full stabilizing effect of the dynamics and 'destabilizing' effect of the noise in that interval, and such a useful estimate is obtainable from our results. Also, the likely escape routes are also of interest, and are obtainable as the minimizers in (1.4) below.

Our rate estimate takes the following form. Let D denote a compact set in the domain of attraction of a stable point θ of the ODE and with $\theta \in D^0$, the interior of D. Let $\delta > 0$ be given. Let $A_D(T)$ denote the set of continuous functions $\phi(\cdot)$ with $|\phi(0) - \theta| \leq \delta$ and $\phi(t) \notin D$ for some $t < T$. We will exhibit a function $\bar{L}(\phi,\dot{\phi},t) \geq 0$ which is zero iff $\dot{\phi} \equiv \bar{b}(\phi)$ and a function $\bar{S}(T,\phi)$:

$$\bar{S}(T,\phi) \equiv \int_0^T \bar{L}(\phi(s),\dot{\phi}(s),s)ds \qquad \text{(for } \phi \text{ absolutely continuous)}$$
$$= \infty \qquad \text{(otherwise)}$$

such that

(1.4) $$\overline{\lim_{n\to\infty}}\ a_n \log P\{X_m \notin D, \text{ some } m \geq n \,|\, |X_n - \theta| \leq \delta\} \leq -\inf_{\substack{\phi \in A_D(T) \\ T>0}} \bar{S}(T,\phi) < 0.$$

The right hand side of (1.4) can yield estimates that are very useful for a 'rate' of convergence, and for the dependence of this rate on the behavior of the algorithm in the set of interest D, as well as for the comparison of algorithms.

In [9], [10], [11], sharp upper and lower bounds were obtained for SA algorithms by the methods of large deviations theory, and a great deal of useful information was presented concerning the bounds and the structure of the H and L-functionals. These references required $a_n \to 0$ in special ways, the noise was 'exogenous', and the dynamical term b was a smooth function of x. The methods were unable to handle the constrained problems. Strictly speaking, the results in these references were not w.p.1 convergence results. They dealt with the sequences of sequences $\{X_m^n, m \geq 0\}$, $n = 1,2, \dots$, defined by $X_{m+1}^n = X_m^n + a_{n+m}b(X_m^n,\xi_{n+m})$, $X_0^n = x$. Although the analysis of such processes is basic to the convergence result, we deal here with the actual process itself. Also, since we are concerned with upper (large deviations) bounds only, we use $\overline{\lim}$ to define the various functionals, rather than lim as illustrated in the sequel. This allows a result under weaker conditions on the $\{a_n,\xi_n\}$, b_n, as will be seen below.

The basic assumptions are stated in Section 2, and examples given to illustrate some of them. The properties of the 'upper bound' which we use instead of the usual log of the exponential moment (the H-functional) are discussed. Some of the conditions ((A2.3) and (A2.6)) are stated in a fairly general form, since they allow a simple proof of the main convergence result, Theorem 3.1, not cluttered with all the details required for all the special cases. It also facilitates the application of future results in large deviations theory to the stochastic approximation problem. In the sequel, we give considerable detail on verifiable sufficient conditions for these assumptions. (A2.3) is a standard assumption in large deviations theory (see also the remarks concerning it in Sections 2 and 6), and it seems to be satisfied in all the examples of interest. Assumption (A2.6) is of a 'large deviations' type itself, and the bulk of the paper is actually devoted to sufficient conditions for it in 'non-smooth cases' (Section 7), constrained and state-dependent noise (Section 5), smooth dynamics and exogeneous noise cases (Section 4).

The total picture is a w.p.1 convergence result with the associated 'escape' probability estimates under quite general conditions.

2. BACKGROUND AND BASIC ASSUMPTIONS

In this section, we introduce some rather general assumptions which will be used to prove the main convergence theorem in Section 3. Two of the assumptions ((A2.3) and (A2.6)) are not easily verifiable, but are used simply to facilitate the proofs in Section 3. We prefer to work with these assumptions in this section, since the conditions and methods which guarantee them differ from case to case. We will return to them in Sections 4 and 5, where readily verifiable sufficient conditions for them are given for a number of cases that cover a wide variety of applications.

Until Section 5, we work only with (1.1), the unprojected case. We say that $\{\xi_n\}$ is 'exogeneous' or 'non-state dependent' if for any n and Borel set A $\in \sigma(\xi_i, i > n)$, we have $P(A \mid \xi_i, i \leq n) = P(A \mid \xi_i, X_i, i \leq n)$. For the 'state-dependent' noise case, we use the model where the pair (X_n, ξ_{n-1}) is a Markov process. This covers a large number of important applications, and provides for a convenient analysis. For the state-dependent case, define the one step transition function (2.1), which we suppose to be independent of n.

$$(2.1) \qquad P^x(\xi,A) = P\{\xi_n \in A \mid X_n = x, \xi_{n-1} = \xi\}.$$

In the state-dependent case, a so-called 'fixed-x' process $\{\xi_n^x\}$ appears in the analysis, exactly as for the weak convergence approach [5]. For each $x \in R^r$, define $\{\xi_n^x\}$ as the M-valued markov process whose transition function is obtained by convolving $P^x(\xi,A)$.

We next define the 'large deviations' H-functional. For the case of exogeneous noise, define $\mathcal{F}_n = \sigma(\xi_i, i \leq n)$ and let $E_{\mathcal{F}_n}$ denote the expectation conditioned on $\mathcal{F}_n$. For the state-dependent noise case, let E^x_ξ den... expectation given $\xi^x_0 = \xi$. We first define the functionals for the case of constant gain $a_n = a > 0$, and then make the alterations which are required when $a_n \to 0$. The following assumptions will be used. Sufficient cond... are given in the remarks following, and in Appendix 1.

A2.1. *Exogeneous noise. The* lim sup *exists uniformly in* ω *and* x *(in any compact set)**

$$\text{(2.2a)} \qquad H(x,\alpha) = \overline{\lim_{N,n}} \frac{1}{n} \log E_{\mathcal{F}_N} \exp\langle\alpha, \sum_{N+1}^{N+n} b_i(x,\xi_i)\rangle$$

State dependent noise case. The lim sup *exists uniformly in* $\xi \in M$, *and* x *(in any compact set)*

$$\text{(2.2b)} \qquad H(x,\alpha) = \overline{\lim_{N,n}} \frac{1}{n} \log E^x_\xi \exp\langle\alpha, \sum_{N+1}^{N+n} b_i(x,\xi^x_{i-N})\rangle$$

A2.2. *There is a continuous function* $\bar{b}(\cdot)$ *such that (exogeneous noise and uniformly in* x *in any compact set and in* ω*) as* $n,N \to \infty$,

$$\text{(2.3a)} \qquad \frac{1}{n}\sum_{N+1}^{N+n} E_{\mathcal{F}_N} b_i(x,\xi_i) \to \bar{b}(x)$$

(state dependent noise, and uniformly in x *in any compact set and in* $\xi \in M$*)*

$$\text{(2.3b)} \qquad \frac{1}{n}\sum_{N+1}^{N+n} E^x_\xi b_i(x,\xi^x_{i-N}) \to \bar{b}(x)$$

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*We say that $\overline{\lim}$ exists uniformly in ω if for any $\delta > 0$ there are N_δ, n_δ such that for $n \geq n_\delta$ and $N \geq N_\delta$, the r.h.s. of (2.2a) is $\leq H(x,\alpha) + \delta$ w.p.1.

[illegible]

Remark. In the [illegible] Freidlin [illegible] in a slightly different [illegible] discrete parameter [illegible]

[illegible]

It is generally [illegible] hard [illegible] expectation [illegible] with the [illegible] requirement that the [illegible] separate [illegible] each interval [illegible] which the [illegible] and [illegible] constant [illegible] conditioning on the data [illegible] the [illegible] interval [illegible] and [illegible] needed [illegible] with (A2) although the entire development [illegible] assumption used by Freidlin.

For each N, n and x (resp. N, n, t and x in 2.2b) the functions

[illegible]

are convex in α and take value [illegible] α = [illegible]. Thus H [illegible] have these properties.

Remark. Since the b_i are bounded, a condition equivalent to (2.2a) (with a similar change for (2.2b)) is to use $E_{\mathcal{F}_{n-m}}$ in lieu of $E_{\mathcal{F}_n}$, where $m = o(n)$.

We use the following assumption on $H(\cdot,\cdot)$ and comment on it below and in Appendix 1.

Define the dual function or Legendre transform

$$L(x,\beta) = \sup_{\alpha} [\langle\alpha,\beta\rangle - H(x,\alpha)]$$

A2.3. (a) $L(\cdot,\cdot)$ *is lower semicontinuous in both variables.* (b) *For each* x, *the gradient* $H_\alpha(x,\alpha)$ *exists at* $\alpha = 0$.

The α-differentiability is a rather weak requirement. Some sufficient conditions are given in the remark below. A more general approach appears in Appendix 1 (Section 6). As discussed in Section 6, it is equivalent to the condition that $L(x,\beta) = 0$ iff $\beta = \bar{b}(x)$, *the mean value of the dynamics*, and seems to be satisfied in all examples of interest.

Remark on the l.s.c. of $L(\cdot,\cdot)$ in (A2.3). The l.s.c. property holds if $H(\cdot,\cdot)$ is continuous. Although conditions guaranteeing this continuity may vary from case to case, it is often quite easy to prescribe mild sufficient conditions for a given case. For example, if $b_i(x,\xi) = b(x,\xi)$, and if $b(x,\xi)$ is continuous in x (uniformly in ξ) then $H(x,\alpha)$ is continuous. Even if $b(\cdot,\xi)$ is not continuous, it is often true that the noise provides enough 'smoothing' so that for some $m > 0$ (not depending on x, ω, N or n) the functions

$$D_{N,n}(x,\alpha) = E_{\mathcal{F}_{N-m}} \exp \sum_{i=N+1}^{N+n} \langle\alpha, b_i(x,\xi_i)\rangle$$

are continuous in (x,α) uniformly in the other variables. Under a mild additional condition, this continuity will give us the l.s.c. property. First, we show it for the stationary m-dependent case, where for any j, $\{\xi_i, i \le j\}$, $\{\xi_i, i > j + m\}$ are mutually independent.

Define $D_q(x,\alpha) = E \exp \Sigma_0^{q-1}\langle\alpha, b_i(x,\xi_i)\rangle$ and $H_q(x,\alpha) = (\log D_q(x,\alpha))/(q+m)$. Suppose that D_q is continuous for each q. By the m-dependent property and the stationarity,

$$\frac{1}{kq + km} \log E_{\mathcal{F}_{-m}} \exp \sum_{1}^{kq+km} \langle\alpha, b_i(x,\xi_i)\rangle$$

$$\le \frac{1}{kq + km} \log[\exp|\alpha| Kkm] \prod_{\ell=1}^{k} E_{\mathcal{F}_{\ell q-m}} \exp \sum_{\ell q}^{\ell q+q-1} \langle\alpha, b_i(x,\xi_i)\rangle$$

$$\equiv \delta_q|\alpha| + H_q(x,\alpha).$$

where $\delta_q = K/(q + m) \to 0$ as $q \to \infty$. Thus $H(x,\alpha) \le H_q(x,\alpha) + \delta_q|\alpha|$.

To show the l.s.c. property of L, proceed as follows. Let $\beta_i \to \beta$, $x_i \to x$ and write

$$\underline{\lim}_i L(x_i,\beta_i) \ge \underline{\lim}_i \sup_{|\alpha|\le M} [\langle\alpha,\beta_i\rangle - H_q(x_i,\alpha) - \delta_q|\alpha|]$$
$$= \sup_{|\alpha|\le M} [\langle\alpha,\beta\rangle - H_q(x,\alpha) - \delta_q|\alpha|].$$

Now, let $q \to \infty$ (so that $H_q(x,\alpha) + \delta_q|\alpha|$ can be replaced by $H(x,\alpha)$), and then let $M \to \infty$ to get by monotonicity that

$$\underline{\lim}_i L(x_i,\beta_i) \ge \sup_\alpha[\langle\alpha,\beta\rangle - H(x,\alpha)] = L(x,\beta),$$

which is the l.s.c. result.

It is also simple to prove the α-differentiability (A2.3b) for this m-dependent and stationary model (even without the continuity in x). At

$\alpha = 0$, the gradient of $H_q(x,\alpha)$ equals $E\,\Sigma_1^q\, b_i(x,\xi_i)/(q+m)$, which converges to a limit $\bar{b}(x)$ as $q \to \infty$. Since the convex (in α) function $H(x,\alpha)$ is bounded above by the convex functions $H_q(x,\alpha) + \delta_q|\alpha|$, and since $H(x,0) = H_q(x,0) = 0$, the set of subdifferentials of $H(x,\cdot)$ at $\alpha = 0$ is contained in the set of subdifferentials of $H_q(x,\cdot) + \delta_q|\cdot|$ for every q. This latter set converges to the point $\bar{b}(x)$ as $q \to \infty$. Hence $H(x,\cdot)$ has $\bar{b}(x)$ as its unique subdifferential at $\alpha = 0$, which implies that $H_\alpha(x,0)$ exists and equals $\bar{b}(x)$.

A proof similar to that above can be employed to get the l.s.c. of $L(\cdot,\cdot)$ if the $D_{N,n}(\cdot,\cdot)$ are continuous for some $d > 0$ (not depending on N, n, x, ω) and the $\overline{\lim}$ in (A2.1) is attained in the following uniform way: Let there be $\delta(N_0,N_1,n_0,n_1)$ (which do not depend on x, ω or α) which goes to zero as $N_1 - N_0 \to \infty$, $n_1 - n_0 \to \infty$, $n_0 \to \infty$, $N_0 \to \infty$ and such that

$$|H(x,\alpha) - \sup_{\substack{N_1 \geq N \geq N_0 \\ n_1 \geq n \geq n_0}} \frac{1}{n} \log D_{N,n}(x,\alpha)| \leq \delta(N_0,N_1,n_0,n_1)(|\alpha| + 1).$$

This condition doesn't seem particularly restrictive.

<u>Remark on the calculation of the derivative</u> $H_\alpha(x,0)$ <u>in</u> (A2.3). We show how to calculate the value of the derivative, given that it exists. The derivative plays a crucial role in the sequel, since it defines the 'mean dynamics' for the algorithm (1.1). The following readily verified facts about convex functions will be used to get $H_\alpha(x,0)$ in terms of the statistics of $\{b_n(x,\xi_n)\}$ or $\{b_n(x,\xi_n^x)\}$.

(i) Let $\{f_i(\cdot)\}$ be convex on R^r and satisfy $f_i(0) = 0$. The $\sup_i f_i(\alpha)$ is differentiable at $\alpha = 0$ only if each $f_i(\cdot)$ is and the gradient $f_{i\alpha}(0)$ does not depend on i.

(ii) Let each $f_i(\cdot)$ in (i) be differentiable at $\alpha = 0$ and let $f(\alpha) = \lim_i f_i(\alpha)$ exist. If $f(\cdot)$ is differentiable at $\alpha = 0$, then $f_\alpha(0) = \lim_i f_{i\alpha}(0)$.

Now we use (i), (ii) and the limit assumptions (A2.1), (A2.2) and the differentiability assumption in (A2.3) to calculate $H_\alpha(x,0)$. By definition

$$\overline{\lim_{\substack{N\to\infty \\ n\to\infty}}} = \lim_{\substack{N_0\to\infty \\ n_0\to\infty}} \sup_{\substack{N\geq N_0 \\ m\geq n_0}} .$$

This, together with the above facts and assumptions allows us to calculate $H_\alpha(x,0)$ as follows. Write $H(x,\alpha)$ in the form

$$\overline{\lim_{n,N}} \frac{1}{n} \log E_{\mathcal{F}_N} \exp\langle \alpha, \sum_{N+1}^{N+n} (b_i(x,\xi_i) - E_{\mathcal{F}_N} b_i(x,\xi_i))\rangle + \frac{1}{n} \sum_{N+1}^{N+n} \langle \alpha, E_{\mathcal{F}_N} b_i(x,\xi_i)\rangle$$

The result that $H_\alpha(x,0) = \bar{b}(x)$ follows by noting that the derivative (at $\alpha = 0$) of the terms to the right of the $1/n$ is zero for all n, N and using (A2.2). In fact, what we have *really* shown is that (A2.1) and (A2.3) imply the existence of $\bar{b}(x)$ such that (2.3a) holds. An analogous calculation works for the state-dependent noise case.

<u>Example 1</u>. Consider the simplest case, where $b_i(x,\xi) = b_i(x)$. Then, under (2.3a),

$$\bar{b}(x) = \lim_{n,N} \frac{1}{n} \sum_{N+1}^{N+n} Eb_i(x).$$

If the $b_i(x)$ are identically distributed for each x, then $\bar{b}(x) = Eb_i(x)$ and $H(x,\alpha) = \log E \exp\langle \alpha, b_i(x)\rangle$, and $\overline{\lim} = \lim$ in (A2.1). If the measure induced by $b_i(x)$ on R^r is weakly continuous in x (as is rather common in applications), then $H(\cdot,\cdot)$ is continuous. The rate of convergence estimates for classical stochastic approximation [1] - [3], [6] do not cover this case unless $b_i(\cdot)$ is an appropriately smooth function of x. Thus, even in this simple case, which covers many applications where the b_i involve (e.g.) indicator functions, *we can get a rate estimate unattainable via the classical theory.*

Example 2. The case of Example 1, but where $\{b_i(x)\}$ are not identically distributed. Define $H_i(x,\alpha) = \log E \exp\langle\alpha, b_i(x)\rangle$. Then $\bar{b}(\cdot)$ is still given by (A2.2) and

$$H(x,\alpha) = \overline{\lim_{N,n}} \frac{1}{n} \sum_{N+1}^{N+n} H_i(x,\alpha),$$

which exists and is differentiable at $\alpha = 0$. The noise process here is non-stationary, but we can still get our 'rate' estimate. The example also illustrates that the use of $\overline{\lim}$ rather than lim in (A2.1) is of much more than academic interest. If the measures of the $b_i(x)$ are weakly continuous in x, uniformly in i, then $\bar{b}(\cdot)$ is continuous.

Example 3. Remarks on the use of $\overline{\lim}$ rather than lim in (A2.1). The use of $\overline{\lim}$ is somewhat equivalent to taking a worst case. For example, let $b_n(x,\xi) = b(x) + \xi$, where $\{\xi_n\}$ is a sequence of zero mean mutually independent Gaussian random variables with covariances $\{\Sigma_n\}$. Since

$$\frac{1}{n} \log E \exp\langle\alpha, \sum_{N+1}^{N+n} (b(x) + \xi_j)\rangle = b(x) + \frac{1}{2n} \sum_{N+1}^{N+n} \langle\alpha, \Sigma_j \alpha\rangle,$$

the $\overline{\lim}$ in (A2.1) is just $b(x) + \alpha'\Sigma\alpha/2$, where Σ is the $\overline{\lim}$ of $\frac{1}{n}\sum_{N+1}^{N+n} \Sigma_i$ in the sense of non-negative definite matrices. In many problems, the dynamics are stable enough so that if the noise terms are multiplied by some factor (to take, say, Σ_n to Σ) we still have the required 'stability' to get the desired w.p.1 convergence.

Example 4. Let $b_i(x,\xi)$ be simply a function $b(x,\xi)$ (i.e., a deterministic random field), and consider the case of Markov state-dependent noise, with one-step transition function $P^x(\xi,\cdot)$. Under a uniform (in the initial condition) recurrence condition on fixed x-process $\{\xi_n^x\}$ and continuity of $b(x,\xi)$ in x (uniformly in ξ), the following facts are proved in [24]. Let C(M) denote the continuous real valued functions on M and define an operator mapping $C(M) \to C(M)$ by

$$\hat{P}(x,\alpha)(f)(\xi) = \int_M \exp\langle\alpha,b(x,\psi)\rangle f(\psi)P^x(\xi,d\psi). \tag{2.4}$$

The eigenvalue $\lambda(x,\alpha)$ of $\hat{P}(x,\alpha)$ with the maximum modulus is real, simple and larger than unity for $\alpha \neq 0$. Also $H(x,\alpha) = \log \lambda(x,\alpha)$ and $H(x,\alpha)$ is analytic in α. If the right side of (2.4) is continuous in x for each $f(\cdot)$, α, ξ, then $H(\cdot,\cdot)$ is continuous. Also $\bar{b}(x) = \int b(x,\xi)u^x(d\xi)$, where $u^x(\cdot)$ is the unique invariant measure of $\{\xi_n^x\}$, and $\overline{\lim} = \lim$ in (A2.1).

These various examples can be combined and extended. Other examples are in Sections 5 and 6, and in [11] and [24].

The Limit ODE and Properties of the L-Function. In the expression (2.3), defining the mean 'dynamics' of (1.1), the terms are weighted equally. This corresponds to the case $a_n \equiv a$. We will see in Section 3 that, under a simple 'asymptotic continuity' condition on $\{a_n\}$, $\bar{b}(\cdot)$ also yields the appropriate 'mean' dynamics when $a_n \to 0$. In order to get any sort of useful convergence for $\{X_n\}$, the ODE

$$\dot{x} = \bar{b}(x) \tag{2.5}$$

must have at least one stable point. We assume:

A2.4. *The* ODE (2.5) *has a unique solution for each initial condition and there is a point* θ *which is asymptotically (not necessarily globally) stable in the sense of Liapunov, with domain of attraction* Λ.

A2.4 implies that for any compact $G \subset \Lambda$ and $\delta > 0$ there is $T < \infty$ such that all solutions originating in G are in $N_\delta(\theta)$, a δ-neighborhood of θ, for $t \geq T$.

Recall the definition of the dual of $H(\cdot,\cdot)$:

$$L(x,\beta) = \sup_{\alpha}[\langle\alpha,\beta\rangle - H(x,\alpha)].$$

The following lemma collects several facts concerning $L(\cdot,\cdot)$ which will be needed later.

<u>Lemma 2.1</u>. *Under* (A2.3),

(i) $L(\cdot,\cdot) \geq 0$.

(ii) $L(x,\beta) = 0$ *iff* $\beta = \bar{b}(x)$.

(iii) $L(x,\beta) = \infty$ *if* $|\beta| > K$.

<u>Proof</u>. (i) This follows from $H(x,0) = 0$.

(ii) The convexity in α and the α-differentiability of $H(x,\alpha)$ at $\alpha = 0$ imply that $\bar{b}(x)$ is the only vector in R^r satisfying

$$H(x,\alpha) - \langle\alpha,\bar{b}(x)\rangle \geq 0.$$

The result follows from this.

(iii) Since $|b_i(\cdot,\cdot)|$ is bounded by K, $H(x,\alpha) \leq K|\alpha|$. If $|\beta| > K$, then by taking $\alpha = n\beta$ we see

$$L(x,\beta) \geq n|\beta|^2 - n|\beta|K \to \infty. \qquad \square$$

Define $t_0 = 0$, $t_n = \Sigma_0^{n-1} a_i$ and $m(t) = \max(n\colon\ t_n \leq t)$. We will require the following 'asymptotic continuity' assumption on the sequence $\{a_n\}$.

A2.5. $\displaystyle\lim_{\substack{|t_n - t_m| \to 0 \\ n,m \to \infty}} \frac{a_n}{a_m} = 1$

For every N we define $K_N(s) = a_{m(t_N+s)}/a_N$. If follows from A2.5 that given $\delta > 0$ there is $c(\delta) > 0$ and $N(\delta) < \infty$ such that $N \geq N(\delta)$ and $|t-s| \leq c(\delta)$ imply $|K_N(t) - K_N(s)| \leq \delta$. Define $K(t) = \overline{\lim}\ K_N(t)$. Then A2.5 implies $K(t)$ is continuous and satisfies $0 < K(t) < \infty$ for $0 \leq t < \infty$.

Examples. Let $a_n = 1/n$. Then $m(t_n + s)/n(\exp s) \to 1$ as $n \to \infty$, and $K_N(s) \to \exp -s$. Let $a_n = 1/n^\gamma$, $\gamma \in (0,1)$. Then $m(t_n+s)/(n+sn^\gamma) \to 1$ as $n \to \infty$ and $K_N(s) \to 1$. If $a_n = c/\log n$, then $m(t_n+s)/(n+s) \to 1$ and $K_N(s) \to 1$. In general, if a_n is nonincreasing, then $K(s) \leq 1$.

The H and L-Functionals for Non-Constant $\{a_n\}$. We next define the analog of the $H(x,\alpha)$ for our case of non-constant $\{a_n\}$. Owing to the fact that a_n is not constant, the H and L functionals will depend on time, if $K(t)$ is not equal to unity. Define the 'centered' H-functional

$$H_0(x,\alpha) = H(x,\alpha) - \langle \alpha, \bar{b}(x) \rangle$$

and set

$$\bar{H}(x,\alpha,s) = K^{-1}(s) H_0(x, K(s)\alpha) + \langle \alpha, \bar{b}(x) \rangle. \tag{2.7}$$

The definition (2.7) and (A2.2), (A2.3) imply the differentiability of $\bar{H}(x,\cdot,s)$ at $\alpha = 0$ with $\bar{b}(x) = \bar{H}_\alpha(x,0,s)$ (it will not actually depend on s). Let $\bar{L}(x,\beta,s)$ denote the dual of $\bar{H}(x,\alpha,s)$:

$$\bar{L}(x,\beta,s) = \sup_{\alpha}[\langle\alpha,\beta\rangle - \bar{H}(x,\alpha,s)].$$

Using (2.7), we see that

$$\begin{aligned}\bar{L}(x,\beta,s) &= \sup_{\alpha}[\langle\alpha,\beta\rangle - K^{-1}(s)H_0(x,K(s)\alpha) - \langle\alpha,\bar{b}(x)\rangle] \\ &= K^{-1}(s)L_0(x,K^{-1}(s)K(s)(\beta - \bar{b}(x))) \\ &= K^{-1}(s)L(x,\beta) \quad ,\end{aligned}$$

where

$$L_0(x,\beta) = \sup_{\alpha}\ [\langle\alpha,\beta\rangle - H_0(x,\alpha)] = L(x,\beta + \bar{b}(x)).$$

(A2.3) and Lemma 2.1 then imply that $\bar{L}$ has the following properties:

(i) $\bar{L}(\cdot,\cdot,\cdot) \geq 0.$

(ii) $\bar{L}(x,\beta,s) = 0$ iff $\beta = \bar{b}(x).$

(iii) $\bar{L}(x,\beta,s)$ is jointly l.s.c. in (x,β).

(iv) $\bar{L}(x,\beta,s) = \infty$ if $|\beta| > K$.

We now define a large deviation *action functional* for (1.1). Let C[0,T] denote the space of R^r-valued continuous functions on [0,T]. Then for $\phi \in$ C[0,T], define the functional

(2.8) $$\bar{S}_x(T,\phi) = \int_0^T \bar{L}(\phi(s),\dot{\phi}(s),s)ds$$

if ϕ is absolutely continuous and $\phi(0) = x$, and set $\bar{S}_x(T,\phi) = \infty$ otherwise.

In the sequel, all functionals of the type (2.8) are assumed to take the value $+\infty$ if $\phi(\cdot)$ is not absolutely continuous or $\phi(0) \neq x$.

For purposes of the next assumption and the proof in Section 3, it is convenient to define (for each N and x) the form $\{X_n^{N,x}, n \geq N\}$ of (1.1), which *starts* at time N with initial condition at time N satisfying $X_N^{N,x} = x$ and then

$$(2.9) \qquad X_{n+1}^{N,x} = X_n^{N,x} + a_n b_n(X_n^{N,x}, \xi_n), \quad n \geq N.$$

In the exogenous case, the noise in (2.9) is the same as in (1.1), while in the state dependent case we will specify $\xi_N \in M$. Define the continuous parameter interpolations of the processes (1.1) and (2.9):

$$(2.10) \qquad X(t) = [(t-t_n)X_{n+1} + (t_{n+1}-t)X_n]/a_n, \quad t \in [t_n, t_n+a_n] = [t_n, t_{n+1}].$$

$$(2.11) \qquad X^{N,x}(t) = [(t-(t_n-t_N))X_{n+1}^{N,x} + ((t_{n+1}-t_N)-t)X_n^{N,x}]/a_n, \quad t \in [t_n-t_N, t_{n+1}-t_N].$$

We will use (A2.6) below, given in terms of the processes $X^{N,x}(\cdot)$. The assumption is certainly not readily verifiable, but it allows a general proof of the w.p.1 convergence and the upper bound to the convergence rate given in Section 3. It is convenient to use the condition as it is stated, since it is the key condition in Theorem 3.1, and in different cases, different sets of conditions would have to replace it. In Sections 4 and 5 we devote considerable attention to a series of verifiable conditions for (A2.6), and cover a large number of interesting cases. Let $C_x[0,T]$ denote the set of continuous R^r-valued functions on [0,T] with initial value x, and with the sup norm topology.

A2.6. *Let* $s > 0$, $\delta > 0$, $T > 0$ *and compact* $F \subset \Lambda^0$ (*the interior of* Λ) *be given. Then there is* $N_0 < \infty$ *such that for any* $x \in F$, *any set* $A \in C_x[0,T]$ *satisfying* $\inf_{\phi \in \bar{A}} \bar{S}_x(T,\phi) \geq s$, *and any* $N \geq N_0$, *we have*

$$\text{(2.13a)} \qquad a_N \log P\{X^{N,x}(\cdot) \in A \mid \mathcal{F}_N\} \leq -s + \delta$$

$$\text{(2.13b)} \qquad a_N \log P\{X^{N,x}(\cdot) \in A \mid \xi_N = \xi\} \leq -s + \delta$$

for almost all ω and all $\xi \in M$ in the cases of exogenous noise and state dependent noise, respectively.

Remark. The uniformity of the estimates with respect to ω (respectively ξ) imply that (2.13a) (resp. (2.13b)) continues to hold if we replace N by any stopping time $M \geq N_0$.

Finally, we state the slowest rate at which we can allow $a_n \to 0$.

A2.7. *For every* $\delta > 0$, $\Sigma_n \exp - \delta/a_n < \infty$, $\Sigma a_n = \infty$.

For example, let $a_n = c_n/\log n$, and $c_n \to 0$ with $\Sigma a_n = \infty$. Then (A2.7) holds.

3. THE BASIC CONVERGENCE THEOREM

The following lemma gives several important properties of our action functional.

Lemma 3.1. *Assume* (A2.1) *to* (A2.3). *Then for any* $T > 0$:

(i) $\bar{S}_{\phi(0)}(T,\phi)$ *is* l.s.c. *in* $\phi \in C[0,T]$.

(ii) *For any compact set* $F \subset R^r$, *and any* $\infty > s \geq 0$, *the set*

$$G = \bigcup_{x \in F} \{\phi : \bar{S}_x(T,\phi) \leq s\}$$

is compact.

(iii) $\bar{S}_x(T,\phi) = 0$ *iff* $\dot{\phi} = \bar{b}(\phi)$ (a.s.) *in* $[0,T]$, *and* $\phi(0) = x$.

(iv) *For each* $\epsilon > 0$ *and* $T < \infty$ *there is a* $\delta > 0$ *such that* $|\beta - b(x)| \geq \epsilon$ *implies* $\bar{L}(x,\beta,s) \geq \delta$ *on* $[0,T]$.

Proof. (i) See [14; Theorem 3, Section 9.1.4].

(ii) Recall that for $|\beta| > K$ implies $\bar{L}(x,\beta,s) = \infty$ for all $s \geq 0$. It follows that $\phi \in G$ implies that ϕ is Lipschitz continuous with constant $\leq K$. Ascoli's theorem then implies that G is precompact, and (ii) now follows from (i).

(iii) $\bar{S}_x(T,\phi) = 0$ iff $\bar{L}(\phi(s),\dot{\phi}(s),s) = 0$ a.s. in $[0,T]$. Since $\bar{L}(x,\beta,s) = 0$ iff $\beta = \bar{b}(x)$, $\bar{S}_x(T,\phi) = 0$ iff $\dot{\phi} = \bar{b}(\phi)$ a.s.

(iv) It is enough to work with $L(x,\beta)$. Let $x_n \to x$, $\beta_n \to \beta$ such that $|\beta_n - \bar{b}(x_n)| \geq \epsilon > 0$ and $L(x_n,\beta_n) \to 0$. By the l.s.c. properties of L, $\underline{\lim}_n L(x_n,\beta_n) \geq L(x,\beta)$ which equals zero only if $\beta = \bar{b}(x)$. $\square$

We now present the convergence theorem.

<u>Theorem 3.1</u>. *Assume* (A2.3) *to* (A2.7), *and that given some compact neighborhood* $G(\theta)$ *of* θ *such that* $G(\theta) \subset \Lambda^0$ *there is* (a.s.) *a (random) sequence* $\{n_i\}$ *such that* $X_{n_i} \in G(\theta)$.

Then $X_n \to \theta$ w.p.1.

Assume in addition that given $\epsilon > 0$ *there is* $\bar{N} < \infty$ *such that* $a_i/a_N \leq 1 + \epsilon$ *for all* $i \geq N \geq \bar{N}$. *Then*

$$(3.1) \qquad \overline{\lim_N}\, a_N \log P\{X_n \notin G(\theta), \textit{ some } n \geq N \mid |X_N - \theta| \leq \delta\}$$

$$\leq - \inf_{\substack{\phi: |\phi(0) - \theta| \leq \delta \\ \phi(t) \notin G(\theta), \text{ some } t < \infty}} \bar{S}_{\phi(0)}(t, \phi) \equiv -\bar{S}^*$$

<u>Remarks</u>. If not all paths visit some neighborhood of θ infinitely often (i.o.) then we will have $X_n \to \theta$ w.p.1 with respect to those paths which do. It is expected that the recurrence condition would be verified by a weak convergence argument. Under the last assumption of the theorem, $K(t) \leq 1$, which implies $\bar{L}(x,\beta,t) \geq L(x,\beta)$. It is then simple to show (see the arguments below) that for small $\delta > 0$ the r.h.s. of (3.1) is strictly negative. In particular, if a_n is nonincreasing, then $K(t) \leq 1$.

<u>Proof</u>. For $\delta > 0$, let $N_\delta(\theta)$ denote $\{x: |\theta - x| \leq \delta\}$. We will first prove that if $\{X_n\}$ visits $G(\theta)$ infinitely often w.p.1, then $\{X_n\}$ visits $N_\delta(\theta)$ infinitely often w.p.1. We can suppose that $N_{2\delta}(\theta) \subset G(\theta)$.

Owing to the stability assumption (A2.4), there is $T_1 < \infty$ such that if ϕ satisfies $\dot{\phi} = \bar{b}(\phi)$ and $\phi(0) = x \in G(\theta)$ then $\phi(t) \in N_{\delta/2}(\theta)$ for $t \geq T_1$. Define the set of paths

$$A_j = \{ \xi \in \ldots \}$$

We ...

$$\ldots \quad \xi \in A \ldots$$

... contradiction.

Define the event

$$E_n^j = \{ X_n \in G_n \theta \ldots \in \mathcal{N}_k \theta \}$$

We have $E^j = \ldots \in A$. The assumption ... then implies that

$$\sum_n P(E_n^j) \le \sum_n P(E_n^j, X_n \in G_n \theta) < \infty$$

and the Borel-Cantelli lemma gives

$$P(E_n^j \text{ occurs infinitely often}) = 0$$

Since $X(T_1 + t_n)$ [illegible] $X_{m T_1 + t_n} \le (a_{m T_1 + t_n} \vee a_{m T_1 + t_n + 1}) K \to 0$, we conclude that the event $\{X_n \in G(\theta),\ X_{m T_1 + t_n} \notin N_{2\delta}(\theta)\}$ ocurs only finitely often, w.p.1. Thus if $\{X_n\}$ visits $G(\theta)$ infinitely often w.p.1, it must visit $N_{2\delta}(\theta)$ infinitely often w.p.1 for each $\delta > 0$.

Now let $\delta_1 > 0$ be such that $N_{\delta_1}(\theta) \subset A \subset B$. By the stability assumption (A7) there is [illegible] $\delta_1 > \delta_2 > 0$ such that for any $x \in N_{\delta_2}(\theta)$, if ϕ satisfies $\dot\phi = h(\phi)$ and $\phi(0) = x$, then $\phi(t) \in N_{\delta_1/2}(\theta)$ for $t \ge 0$. By the preceding [illegible] we can assume that

$$P\{X_n \in N_{\delta_2}(\theta) \text{ infinitely often}\} = 1.$$

Let $T_2 < \infty$ be such that if ϕ satisfies $\dot\phi = h(\phi)$ and $\phi(0) = x \in N_{\delta_2}(\theta)$, then $\phi(t) \in N_{\delta_2/2}(\theta)$ for all $t \ge T_2$. Define

$$A_2 = \{\phi \in C[0, T_2] : \phi(0) \in N_{\delta_2}(\theta) \text{ and there is}$$
$$t \le T_2 \text{ such that } \phi(t) \notin N_{\delta_1}(\theta) \text{ and/or } \phi(T_2) \notin N_{\delta_2}(\theta)\}$$

By an argument analogous to that used to get (3.2), there is a $c_2 > 0$ such that

$$\inf_{\phi \in A_2} S_\phi(T_2, \phi) = c_2 > 0.$$

Define $E_n^2 = \{X_n \in N_{\delta_2}(\theta)$ and there is $t \le T_2$ such that $X(t + t_n) \notin N_{\delta_1}(\theta)$ and/or $X(T_2 + t_n) \notin N_{\delta_2}(\theta)\}$. Then $E_n^2 = \{X(\cdot + t_n) \in A_2\}$.

Let m_k denote the return times of $\{X_n\}$ to N_{δ_2} and note that the $\delta_1 > 0$ can be made arbitrarily small. Since $m_k < \infty$ for all k w.p.1, to prove the theorem it is sufficient to show that

(3.3) $$\lim_n P\{X_{n+i} \notin N_{\delta_1}(\theta), \text{ some } i < \infty \mid X_n \in N_{\delta_2}(\theta)\} = 0.$$

We have the obvious inclusion

$$\begin{aligned}\{X_{n+i} &\notin N_{\delta_1}(\theta), \text{ for some } i < \infty \text{ and } X_n \in N_{\delta_2}(\theta)\} \\ &\subset \{X_i \notin N_{\delta_1}(\theta) \text{ for some } m(jT_2 + t_n) < i \leq m(jT_2 + T_2 + t_n) \\ &\qquad \text{and/or } X_{m(jT_2+T_2+t_n)} \notin N_{\delta_2}(\theta)\text{, some } 0 \leq j < \infty \text{, and} \\ &\qquad X_n \in N_{\delta_2}(\theta)\} \\ &= \bigcup_{0\leq j<\infty} E^2_{m(jT_2+t_n)} \bigcap_{i<j} (E^2_{m(iT_2+t_n)})^C \cap \{X_n \in N_{\delta_2}(\theta)\} .\end{aligned}$$

It follows that

(3.4) $$\begin{aligned}&P\{X_{n+i} \notin N_{\delta_1}(\theta)\text{, some } i < \infty \mid X_n \in N_{\delta_2}(\theta)\} \\ &\leq \sum_{j=0}^{\infty} P\{E^2_{m(jT_2+t_n)} \mid \bigcap_{i<j} (E^2_{m(iT_2+t_n)})^C \cap \{X_n \in N_{\delta_2}(\theta)\}\} .\end{aligned}$$

Note that for any fixed j inclusion in the conditioning set implies $X_{m(jT_2+t_n)} \in N_{\delta_1}(\theta)$. Thus (3.3) follows from (3.4), (A2.6), and (A2.7).

We now consider (3.1). Let $\bar{T} > 0$ be fixed. Define the set of paths

$$\begin{aligned}A(T) = \{\phi: |\phi(0) - \theta| \leq \delta,\ \phi(t) \notin G(\theta) \textit{ for some } t \leq T, \textit{ and} \\ |\phi(t) - \theta| \geq \delta/2 \textit{ for } \bar{T} \leq t \leq T\}.\end{aligned}$$

We claim that for large enough T,

(3.5) $$\inf_{\phi\in\overline{A(T)}} \bar{S}_{\phi(0)}(T,\phi) \geq \bar{S}^* .$$

First note that the same proof as that of (3.2) implies there is $c_3 > 0$ and $T_3 < \infty$ such that if we define $A_3 = \{\phi : \phi(0) \in G(\theta),\ \phi(T_3) \notin N_{\delta/2}(\theta)\}$, then

$$\inf_{\phi \in \bar{A}_3} \int_0^{T_3} L(\phi(s), \dot{\phi}(s))\, ds \geq c_3 .$$

Let i = the integer part of $(T-\bar{T})/T_3$. Then for the paths in A(T) that do not escape from $G(\theta)$, we have

$$\bar{S}_{\phi(0)}(T,\phi) \geq ic_3 ,$$

which implies (3.5) (when T is large).

Now define the stopping times τ_i^N by $\tau_0^N = N$, $\tau_{i+1}^N = \inf\{n \geq m(\tau_i^N + \bar{T}): X_n \in N_\delta(\theta) \text{ or } X_n \notin G(\theta)\}$ and the events $E_i^N = \{X_{\tau_{i+1}^N} \notin G(\theta) \text{ or } t_{\tau_{i+1}^N} - t_{\tau_i^N} \geq T\}$. We use the following estimate, which is derived in the same way as (3.4).

$$P\{X_n \notin G(\theta),\ \textit{some}\ n \geq N \,|\, X_N \in N_\delta(\theta)\}$$

$$\leq \sum_{j=0}^{\infty} P\{E_j^N \,|\, \bigcap_{i<j} (E_i^N)^C \cap \{X_N \in N_\delta(\theta)\}\} . \tag{3.6}$$

Fix $h_1 > 0$. By (A2.6) and (3.5), an upper bound to the r.h.s. of (3.6) is given by

$$\sum_{i=N}^{\infty} \exp - (\bar{S}^* - h_1)/a_i$$

$$= (\exp - (\bar{S}^* - h_2)/a_N) \sum_{i=N}^{\infty} \exp[-(\bar{S}^* - h_1)/a_i + (\bar{S}^* - h_2)/a_N]$$

when N is large. Thus (3.1) follows if we prove that given $h_2 > 0$ there is $h_1 > 0$, $\bar{N} < \infty$, and $M < \infty$ so that for $N \geq \bar{N}$,

$$(3.7) \qquad \sum_{i=N}^{\infty} \exp[-(\bar{S}^* - h_1)/a_i + (\bar{S}^* - h_2)/a_N] \leq M.$$

To prove (3.7), take $\epsilon = (h_2/8\bar{S}^*) \wedge ½$, and $h_1 = \epsilon\bar{S}^*/2$. Pick $\bar{N}$ large enough so that $a_i/a_N \leq 1 + \epsilon$ for $i \geq N \geq \bar{N}$. Then for i such that $a_i/a_N \geq 1 - \epsilon$ we have

$$[-\bar{S}^* + h_1 + (\bar{S}^* - h_2)a_i/a_N]/a_i$$

$$\leq [h_1 + \epsilon\bar{S}^* - h_2(1 - \epsilon)]/a_i \leq [-h_2/4]/a_i .$$

On the other hand, if $a_i/a_N < 1 - \epsilon$, we obtain the following bound for the exponent:

$$[h_1 - \epsilon\bar{S}^*]/a_i = [-\epsilon\bar{S}^*/2]/a_i .$$

Hence (3.7) follows from (A2.7). $\square$

4. A PROOF OF (A2.6) FOR EXOGENEOUS NOISE AND SMOOTH DYNAMICS

In this section, we prove (A2.6) under more readily verifiable conditions, and in the exogeneous noise case. In Section 5, we state and discuss other sets of conditions under which a similar proof yields (A2.6).

First, we show that the H-functionals as defined by (2.7) are the appropriate ones for the case $a_n \to 0$ in a general setting. Then, in Theorem 4.2, a basic sufficient condition for (A2.6) will be obtained.

<u>Theorem 4.1</u>. *Assume* (A2.1), (A2.2) *and* (A2.5). *Then for the exogeneous noise case and uniformly in* ω (w.p.1.) *and in* t *in any bounded interval,*

$$\text{(4.1a)} \qquad \overline{\lim_{\Delta\to 0}} \frac{1}{\Delta} \overline{\lim_{N}}\, a_N \log E_{\mathcal{F}_N} \exp\left[\sum_{m(t_N+t)}^{m(t_N+t+\Delta)} \langle \alpha, a_i b_i(x,\xi_i)\rangle / a_N\right] \leq \bar{H}(x,\alpha,t).$$

For the state dependent noise case, and uniformly in ξ *and in* t *in any bounded interval*

$$\text{(4.1b)} \qquad \overline{\lim_{\Delta\to 0}} \frac{1}{\Delta} \overline{\lim_{N}}\, a_N \log E_{\xi}^{x} \exp\left[\sum_{m(t_N+t)}^{m(t_N+t+\Delta)} \langle \alpha, a_i b_i(x,\xi_{i-N}^{x})\rangle / a_N\right] \leq \bar{H}(x,\alpha,t).$$

Assume in addition (A2.3). *Then for any* $0 \leq T_1 < T_2 < \infty$, $H_\alpha(x,0,t)$ *equals* $\bar{b}(x)$ *where* $\bar{b}(x)$ *also satisfies*

$$\text{(4.2)} \qquad \bar{b}(x) = \frac{1}{T_2 - T_1} \lim_{N\to\infty} E_{\mathcal{F}_N} \sum_{m(t_N+T_1)}^{m(t_N+T_2)} a_i b_i(x,\xi_i),$$

with an analogous statement holding for the state dependent noise case.

<u>Proof</u>. We only prove (4.1a). The proof of (4.1b) is similar. Also, (4.2) is

obvious by (A2.5). Fix $\Delta > 0$ and define $\bar{b}_i^N(x) = E_{\mathcal{F}_N} b_i(x, \xi_i)$. Rewrite the left side of (4.1a) as

$$
\begin{aligned}
(4.3) \qquad & \overline{\lim_{\Delta}}\ \frac{1}{\Delta}\ \overline{\lim_{N}}\ a_N \log E_{\mathcal{F}_N} \exp\left[\sum_{m(t_N+t)}^{m(t_N+t+\Delta)} \langle \alpha, a_i (b_i(x,\xi_i) - \bar{b}_i^N(x)) \rangle / a_N\right] \\
& \qquad + \lim_{\Delta} \frac{1}{\Delta} \lim_{N} a_N \sum_{m(t_N+t)}^{m(t_N+t+\Delta)} \langle \alpha, a_i \bar{b}_i^N(x) \rangle / a_N
\end{aligned}
$$

Under (A2.2) and (A2.5), the last term in (4.3) equals $\langle \alpha, \bar{b}(x) \rangle$. Thus, we need only work with the first term in (4.3). It will be proved that the first term is bounded above by

$$
(4.4) \qquad K^{-1}(t) H_0(x, K(t)\alpha)
$$

which will yield the theorem in view of the definition $\bar{H}(x,\alpha,t) = K^{-1}(t) H_0(x, K(t)\alpha) + \langle \alpha, \bar{b}(x) \rangle$.

By differentiating the part of the first term of (4.3) to the right of the a_N term with respect to a_i, we see that it is convex in $\{a_i\}$, non-negative and zero if $a_i \equiv 0$. Because of this, the definitions of the $K_N(\cdot)$ (below (A2.5)) imply that there are $c_N(\Delta)$ tending to zero as $N \to \infty$ and then $\Delta \to 0$ such that an upper bound to the first term of (4.3) is obtained by replacing the a_i/a_N there by an upper bound $(K_N(t) + c_N(\Delta))$, and by replacing the left hand a_N by an upper bound $\Delta(K_N^{-1}(t) + c_N(\Delta))/[m(t_N+t+\Delta) - m(t_N+t)]$.

We next make use of the following fact. Given a convex function $H(\alpha)$ such that $H(0) = 0$ and $H(\alpha) \geq 0$, the inequality $H(s\alpha') \leq sH(\alpha')$ is valid for all $0 \leq s \leq 1$, and for all α'. Picking $s = s_1/s_2$ and $\alpha' = s_2\alpha$, we obtain for all $0 \leq s_1 \leq s_2$ and for all α that

(4.5) $\qquad s_1^{-1}H(s_1\alpha) \leq s_2^{-1}H(s_2\alpha)$.

We will take $s_1 = K_N(t) + c_N(\Delta)$ and $s_2 = K(t) + c_N(\Delta) + \Delta$.

Since $K_N(t) \leq K(t) + \Delta$ for large N, an upper bound to the first term in (4.3) is also obtained by replacing the a_i/a_N by $(K(t) + c_N(\Delta) + \Delta)$ and the a_N by

$$\frac{\Delta(K_N^{-1}(t) + c_N(\Delta))(K_N(t) + c_N(\Delta))}{(K(t) + c_N(\Delta) + \Delta)(m(t_N + t + \Delta) - m(t_N + t))} .$$

Doing the substitution, using the definitions of $\bar{b}(x)$ and $H(x,\alpha)$ and taking limits yields the desired bound (4.4) for the first term in (4.3) $\square$

Remark. We have used the fact that A2.5 implies $K_N^{-1}(t)$ is bounded from above uniformly in N for N large.

In Theorem 4.2, we prove (A2.6) under condition (A4.1) below.

Remark. For a continuous parameter problem in [12], Freidlin uses a continuous parameter analog of (A4.1), with Lipschitz continuity of $b(\cdot,\xi)$ and continuity and α-differentiability of $H(\cdot,\cdot)$ to get the large deviations inequalities. He uses lim rather than $\overline{\lim}$ to define his H-functional. An examination of the proof in [12] shows that (uniform) continuity of $b(\cdot,\xi)$ is enough. Also, for our 'upper bounding' needs the $\leq$ in the $\overline{\lim}$ of (A2.1) is enough. It's not actually necessary that the lim exists.

A4.1. $\{\xi_n\}$ *is exogeneous. The random vector field* $b_n(x,\xi)$ *is deterministic - and so we write it as* $b(x,\xi)$, *where* $b(\cdot,\xi)$ *is continuous, uniformly in* ξ *and* $|b(x,\xi)| \leq K < \infty$.

Theorem 5.1 extends Theorem 4.2 to the 'non-deterministic' random field case.

Theorem 4.2. (A2.6) *holds under* (A2.1), (A2.3), (A2.5) *and* (A4.1).

Remark. The proof is along the lines of Freidlin, Theorem 2.1 of [12], with appropriate modifications for our use of $\overline{\lim}$, $a_n \to 0$ and the uniformity in x required in (A2.6). We will use the results in Freidlin's proof whenever possible to simplify our argument.

Proof. (i) Since $b(x,\xi)$ is continuous in x, the $\overline{\lim}$ defining H and $\bar{H}$ and $\bar{b}$ are taken on uniformly in x in any compact set, and the $\overline{\lim}$ in (4.1a) also holds uniformly in x (and also in ω, w.p.1). This uniformity implies the following. Let $T < \infty$. Let $F \subset R^r$ be compact and let $\Delta > 0$. Let $\alpha(\cdot)$ and $\psi(\cdot)$ be functions defined on [0,T] that are constant on intervals of the form $[i\Delta, i\Delta+\Delta)$, and let $\psi(\cdot)$ be F-valued. (Assume w.l.o.g. that T is an integral multiple of Δ.) Then, uniformly in $\psi(\cdot)$ and ω,

$$\overline{\lim_N}\, a_N \log E_{\mathcal{F}_N} \exp\left[\sum_N^{m(t_N+T)} \langle \alpha(t_i - t_N), a_i b(\psi(t_i - t_N), \xi_i)\rangle / a_N\right] \le \int_0^T \bar{H}(\psi(t), \alpha(t), t)\,dt. \tag{4.6}$$

(ii) For fixed x and the above defined $\psi(\cdot)$, define the process $\{X_n^{\psi,N}\}$, analogously to the definition of $\{X_n^{x,N}\}$ by $X_N^{\psi,N} = x$ and

$$X_{n+1}^{\psi,N} = X_n^{\psi,N} + a_n b(\psi(t_n - t_N), \xi_n), \tag{4.7}$$

and its piecewise linear version (analogous to the definition of $X^{x,N}(\cdot)$)

$$X^{\psi,N}(t) = [(t - (t_n - t_N))X^{\psi,N}_{n+1} + ((t_{n+1} - t_N) - t)X^{\psi,N}_n]/a_n \quad \text{for} \quad t \in [t_n - t_N, t_{n+1} - t_N). \tag{4.8}$$

The process $X^{\psi,N}(\cdot)$ plays an important intermediary role in getting the desired large deviations result, since it is relatively easy to get one for $X^{\psi,N}(\cdot)$, and then to extend it by suitable choices of $\psi(\cdot)$.

Define $X^{\psi,\Delta}_N = (X^{\psi,N}(i\Delta),\ i = 1, \ldots, T/\Delta)$. We next prove a large deviations upper bound for the *vector* $X^{\psi,\Delta}_N$, which will be uniform in x (in any compact set and also in ω, w.p.1). Let $F_1 \subset R^r$ be compact. Let $\alpha_i \in R^r$, $i \leq T/\Delta$ and define $\alpha(\cdot)$ by (the manipulations at this point are similar to those used in [12, Lemma 3.1])

$$\alpha(s) = \sum_{i=k}^{T/\Delta} \alpha_i, \quad s \in [k\Delta - \Delta, k\Delta).$$

Then by (4.6) (where the $\overline{\lim}$ is uniform in $x \in F_1$, ω (w.p.1.) and $\psi(\cdot)$)

$$\begin{aligned} &\overline{\lim_N}\, a_N \log E_{\mathcal{F}_N} \exp\left[\sum_i^{T/\Delta} \langle \alpha_i, X^{\psi,N}(i\Delta)\rangle / a_N\right] \\ &= \overline{\lim_N}\, a_N \log E_{\mathcal{F}_N} \exp\left[\sum_N^{m(t_N+T)} \langle \alpha(t_i - t_N), a_i(b(\psi(t_i - t_N), \xi_i) + x)\rangle / a_N\right] \\ &\leq \int_0^T \bar{H}(\psi(t), \alpha(t), t)dt + \langle x, \sum_1^{T/\Delta} \alpha_i\rangle \\ &= \sum_0^{T/\Delta - 1} \Delta \bar{H}\left\{\psi(i\Delta), \sum_{j=i+1}^{T/\Delta} \alpha_j, i\Delta\right\} + \langle x, \sum_1^{T/\Delta} \alpha_i\rangle \\ &\equiv h^{x,\psi}\ (\alpha_1, \ldots, \alpha_{T/\Delta}) \end{aligned} \tag{4.9}$$

For $\{\beta_i,\ i \leq T/\Delta\} \equiv \tilde{\beta} \in (R^r)^{T/\Delta}$, define $\ell^{x,\psi}(\beta_1, \ldots, \beta_{T/\Delta})$ to be the Legendre transform of $h^{x,\psi}(\alpha_1, \ldots, \alpha_{T/\Delta})$.

Remark. The last equality in (4.9) is not really correct, since $\bar{H}(\psi,\alpha,t)$ may differ from $\bar{H}(\psi,\alpha,i\Delta)$ over the interval $[i\Delta, i\Delta + \Delta]$. However, since the Legendre transform of $\bar{H}(\psi,\alpha,s)$ is $K^{-1}(s)L(\psi,\beta)$, neglecting this variation amounts to no more than multiplying $\ell^{x,\psi}(\bar{\beta})$ by a scale factor which tends to one as Δ tends to zero, uniformly in all the other variables $(x,\psi,\bar{\beta})$. Since we are subsequently allowed to choose $\Delta > 0$ as small as desired, we can safely ignore the time variations over the interval $[i\Delta, i\Delta + i\Delta]$ as a matter of notational convenience. We maintain this convention in later proofs as well but will use $\approx$ or $\lesssim$ rather than $=$ or $\leq$ to indicate that we are ignoring such a scale factor.

Define $\tilde{\Phi}_x^{\psi,\Delta}(s) = \{\bar{\beta}: \ell^{x,\psi}(\bar{\beta}) \leq s\}$. Then (4.9) and a theorem of Gärtner's ([16], Lemma 1.1) imply that for any $\delta > 0$, $h > 0$ there is a $N_0 < \infty$ such that $N \geq N_0$ implies that (for $x \in F_1$, $\psi(\cdot)$ as above)

(4.10) $$P_{\mathcal{F}_N}\{d_1(X_N^{\psi,\Delta}, \tilde{\Phi}_x^{\psi,\Delta}(s)) > \delta\} \leq \exp - (s-h)/a_N.$$

Here d_1 is the Euclidean metric on $(R^r)^{T/\Delta}$.

In the proof of his result, Gärtner used a definition of (his) H-functional (it is the function G in (1.1) in [16]) which involved a lim rather than a $\overline{\lim}$. But the proof of his Lemma 1.1 is valid if $\overline{\lim}$ is used or any upper (l.s.c.) bound to the $\overline{\lim}$ is used, if that upper bound is used to compute the L-functional. Also, according to the proof in [16], the inequality (4.10) is valid uniformly in all variables in which the inequality (4.1a) is attained uniformly as $N \to \infty$, $\Delta \to 0$. Hence (4.10) holds for a.a. ω, all $x \in F_1$, and $\psi(\cdot)$ as above.

(iii) From this point on the details are essentially the same as for the classical case [12, Theorem 2.1] (which also uses Gartner's result (4.10) for the 'classical' case), and only an outline will be given. The interested reader should refer to [12] to fill in the gaps. A main difference is that we must be more careful about the uniformity of the estimates in x and ω. The argument can be divided into the following steps.

(a) From the definitions of $\ell^{x,\psi}$ and $\bar{L}$, it can be shown [12, Lemma 3.1, p. 137] that

$$\ell^{x,\psi}(\bar{\beta}) \approx \int_0^T \bar{L}(\psi(s), \dot{\beta}(s), s)ds$$

where we define $\beta(\cdot)$ by the linear interpolation

$$\beta(s) = [(i\Delta + \Delta - s)\beta_i + (s - i\Delta)\beta_{i+1}]/\Delta, \quad \text{for } s \in [i\Delta, i\Delta + \Delta].$$

(b) Since $|b(x,\xi)| \leq K$, the $X^{\psi,N}(\cdot)$ are Lipschitz continuous with constant K. Since

$$\inf_{|\beta|>K} \bar{L}(x,\beta,t) = \infty$$

for all x, the paths in the sets

$$\bar{\Phi}_x(s) = \{\phi: \bar{S}_x(T,\phi) \leq s\}$$

and

$$\bar{\Phi}_x^{\psi,\Delta}(s) = \left\{\phi: \int_0^T \bar{L}(\psi(t), \dot{\phi}(t), t)dt \leq s\right\}$$

are also Lipschitz continuous with constant K.

These facts imply that given $\delta > 0$, there are $\Delta_0 > 0$ and $\delta' > 0$ such that for $\Delta \leq \Delta_0$ and all x (d and d_1, resp., are the sup norm and Euclidean distances)

$$\{d(X^{\psi,N}, \bar{\Phi}_x^{\psi,\Delta}(s)) > \delta\} \subset \{d_1(X_N^{\psi,\Delta}, \tilde{\Phi}_x^{\psi,\Delta}(s)) > \delta'\}.$$

This and (4.10) imply that given $h > 0$ and $\delta > 0$ there is $N_0 < \infty$ such that for $N \geq N_0$, a.a. ω, $x \in F_1$ and ψ as above we have

(4.11) $$P_{\mathcal{F}_N}\{d(X^{\psi,N}, \bar{\Phi}_x^{\psi,\Delta}(s)) > \delta\} \leq \exp - (s-h)/a_N .$$

(c) As a consequence of the l.s.c. property (as a function of $\psi(\cdot)$, $\phi(\cdot)$) of $\int_0^T \bar{L}(\psi(s),\dot{\phi}(s),s)ds$, given $h > 0$, there is $\delta_1 > 0$ such that if $d(\psi,\phi) \leq \delta_1$ and $x = \phi(0) \in F_1$ and $\bar{S}_x(T,\phi) \geq s$, then $\int_0^T \bar{L}(\psi(s),\dot{\phi}(s),s)ds \geq s - h$ [12; p. 142].

(d) Since $b(\cdot,\xi)$ is continuous, uniformly in ξ, given $h > 0$ and $\delta_1 > 0$ (as in (c)) and $\delta_2 > 0$, there is a $\delta > 0$ and $\delta_1' > 0$ (and $\leq \delta_1$) such that $\phi \in C_x[0,T]$, $d(\phi,\psi) \leq \delta_1'$ implies that

(4.12) $$\{d(X^{N,x},\phi) \leq \delta\} \subset \{d(X^{\psi,N},\phi) \leq \delta_2\}.$$

In [12], Freidlin uses a Lipschitz condition on $b(\cdot,\xi)$ to get the set inclusion analogous to (4.12). But continuity is also sufficient.

(c) We now combine the facts in (a) - (d). Let $h > 0$ be given and define δ_2, δ_1', δ as in part (d). Set $\hat{\delta} = \min[\delta_2,\delta,\delta_1']/2$. Define the compact set $R(x) = \{\phi \in C_x[0,T]: \phi$ is Lipschitz continuous with constant $K\}$. Let $\{\phi_i, i \leq M\}$ be a $\hat{\delta}$-net of $R(0)$. Then $\{\phi_i^x = x + \phi_i, i \leq M\}$ is a $\hat{\delta}$-net of $R(x)$. Choose $\Delta > 0$ and ψ_i such that the ψ_i are constant on the intervals $[j\Delta, j\Delta+\Delta)$, $j \leq T/\Delta$, and $\sup_i d(\phi_i,\psi_i) \leq \hat{\delta}/2$. Define $\psi_i^x = x + \psi_i$. For x ranging over F_1, the $\psi_i^x(t)$, $t \leq T$ take values in some compact set.

By the set inclusion in (4.12) and the definition of $\hat{\delta}$, we have

$$(4.13)\qquad P_{\mathcal{F}_N}\{d(X^{N,x}, \bar{\Phi}_x(s)) > \delta\}$$

$$\leq \sum_{1}^{M} P_{\mathcal{F}_N}\{d(X^{N,x}, \phi_i^x) \leq \hat{\delta}\} I_{\{d(\phi_i^x, \bar{\Phi}_x(\cdot)) > \hat{\delta}\}}$$

$$\leq \sum_{1}^{M} P_{\mathcal{F}_N}\{d(X^{\psi_i^x, N}, \phi_i^x) \leq \delta_2\} I_{\{d(\phi_i^x, \bar{\Phi}_x(\cdot)) > \hat{\delta}\}}.$$

If $d(\phi, \bar{\Phi}_x(s)) \geq 0$, then $\bar{S}_x(T,\phi) \geq s$. It follows from part (c) that $I_{\{d(\phi_i^x, \bar{\Phi}_x(s)) > \hat{\delta}\}} = 1$ implies

$$(4.14)\qquad \{d(X^{\psi_i, N}, \phi_i^x) \leq \delta_2\}$$

$$\subset \left\{X^{\psi_i^x, N} \in \left[\phi\colon \int_0^T \bar{L}(\psi_i^x(t), \dot{\phi}(t), t)dt \geq s - h,\ \ \phi(0) = x\right]\right\}.$$

Now, by part (b) there is $N_0 < \infty$ (not depending on $x \in F_1$) such that for $N \geq N_0$ and a.a. ω,

$$(4.15)\qquad P_{\mathcal{F}_N}\{d(X^{\psi_i^x, N}, \phi_i^x) \leq \delta_2\} I_{\{d(\phi_i^x, \bar{\Phi}_x(\cdot)) > \hat{\delta}\}} \leq \exp - (s - 2h)/a_N.$$

Combining (4.13) and (4.15) yields that there is $N_0 < \infty$ (not depending on $x \in F$ or on ω (w.p.1)) such that

$$(4.16)\qquad P_{\mathcal{F}_N}\{d(X^{N,x}, \bar{\Phi}_x(s)) > \delta\} \leq \exp - (s - 3h)/a_N.$$

Now suppose that we are given $A \subset C_x[0,T]$ satisfying $\inf_{\phi \in A} \bar{S}_x(T,\phi) \geq s$. We claim that $d(A, \bar{\Phi}_x(s-h)) > 0$. If not, there are $\phi_i^1 \in A$, $\phi_i^2 \in \bar{\Phi}_x(s-h)$ such that $d(\phi_i^1, \phi_i^2) \to 0$. Since $\bar{\Phi}_x(s-h)$ is compact, we can assume that $\phi_i^2 \to \phi \in \bar{\Phi}_x(s-h)$. Then $\phi_i^1 \to \phi$ implies that $\phi \in \bar{A}$. By the l.s.c. of $\bar{S}_x(T,\cdot)$, we have $\bar{S}_x(T,\phi) \leq \underline{\lim}_i \bar{S}_x(T,\phi_i^2) \leq s - h$, a contradiction. It follows that there is $\delta > 0$ such that $\inf_{\phi \in A} \bar{S}_x(T,\phi) \geq s$ implies that $d(\bar{A}, \bar{\Phi}_x(s-h)) > \delta$ for all such A. Together with (4.16) (with s replaced by $s - h$ there), this yields the existence of $N_0 < \infty$ (not depending [illegible] $\in$ [illegible] the [illegible] set A [illegible]

(4.[illegible]) $\quad P_{T}(X^{N} \in A) \le \exp($[illegible]$4h$[illegible]$)$

This [illegible] $\Box$

5. EXTENSIONS

5.1 A 'Mollifier' Method for Non Smooth Random Fields

Section 4 dealt with a process where the (deterministic random field) [illegible] The [illegible] was essential for the extension of [illegible] to get large deviation results for the $X^{\epsilon,N}$ [illegible] the X process. It is often possible to alter the system slightly so that the dynamics [illegible] and the large deviations estimates are altered only by an [illegible] amount. Then the method of Section 4 can be used to get [illegible] 4. We state the general condition and then [illegible] the [illegible] continuous then Theorem 5 extends Theorem [illegible] deterministic random field case.

A5 [illegible] ξ [illegible]

[illegible] $h_i(x, \xi)$ are bounded [illegible] K [illegible] $h_i(x, \xi)$ on R^r [illegible] uniformly [illegible] ξ.

We retain the definitions of H, $\bar{H}$, L and $\bar{S}_x$ from the previous sections.

<u>Theorem 5.1</u> *Under (A5) and (A2.1)–(A2.3), (A2.5), condition (A2.6)* [illegible]

<u>Discussion of the Proof</u>. First the $h(x, \xi)$ are perturbed by ϵ_i where $\{\epsilon_i\}$ is a sequence of zero mean normally distributed random variables with covariance $\sigma^2 I$ and which alter the paths by a very small amount with a very high probability for small σ. Then a sequence v_n of i.i.d. random variables is introduced so that the random field $h_n(x_n, \xi_n) + \epsilon_n$ can be represented (in the

sense that the resulting processes have the same measures) as $F(x_n, \xi_n, v_n)$, where F is continuous in x. Then, a proof very close to that of Theorem 4.2 is used. The details for a *scalar case* (for notational simplicity) where $b_i(x,\xi) = b_i(x)$ are given in Appendix 2. They are an adaptation of the proof of the vector case large deviations upper bound given in [17] for the constant $a_n \equiv \epsilon$ case. An analogous adaptation for the general vector case yields Theorem 5.1.

5.2. State Dependent Noise

We now prove the form of Theorem 4.2 for the state dependent noise case. In order to provide a reasonably general proof, we generalize slightly the definition of the state dependent $\{\xi_n^x\}$ or Markov $\{\xi_{n-1}, X_n\}$ processes. Let $\{x_n\}$ be a sequence of either random variables and/or constants and $\{\xi_n\}$ a random sequence with the following properties. x_n is a function (perhaps deterministic) of $\{x_j, \xi_j, j < n\}$, and $P\{\xi_n \in \cdot \mid \xi_{n-1} = \xi, x_{n-1} = x, \xi_{n-i}, x_{n-i}, i < n\} = P^x(\xi, \cdot)$. Such an $\{x_n\}$ sequence is said to *generate* $\{\xi_n\}$. Most often $x_n = X_n$ or $X_n^{N,x}$ for some (to be stated where necessary) initial condition and starting time.

We use the following condition, where $H(x,\alpha)$ was defined in (2.2b). The condition is satisfied in many problems of practical interest. An example will be given at the end of the subsection.

A5.2. (i) $\{b_i(\cdot,\cdot)\}$ *is i.i.d. and* $|b_i(x,\xi)| \leq K$.

(ii) *Given* $\gamma > 0$, *there is a* $\delta > 0$ *such that if* $\{x_i\}$ *generates* $\{\xi_i\}$ *and* $|x_i - x| \leq \delta$ *for all* i, *then* (E_ξ *denotes the expectation given the initial condition* $\xi_0 = \xi$)

$$(5.1) \qquad \overline{\lim}\, \frac{1}{N} \log E_\xi \exp \langle \alpha, \sum_1^N b_i(x_i, \xi_i) \rangle \leq H(x,\alpha) + \gamma(|\alpha| + 1) \equiv H_\gamma(x,\alpha),$$

uniformly in $\xi \in M$ *and in* x *in any compact set.*

<u>Theorem 5.2</u>. *Under* (A5.2) *and* (A2.1), (A2.3), (A2.5), *condition* (A2.6) *holds.*

<u>Proof</u>. The proof will be set up so that it can be completed by an argument of the type used in Sections (c) and (e) of Theorem 4.2. The basic

technique is adapted from [20], where a general treatment of the upper and lower large deviations bounds are obtained for the constant $a_n \equiv a > 0$ case

Fix $\gamma > 0$, and let δ be defined by (A5.2). The proof of Theorem 4.1 adapted to condition (A5.2) yields: for any sequence $\{x_i\}$ generating $\{\xi_i\}$ and satisfying $|x_i - x| \le \delta$, we have (uniformly in ξ and in x in any compact set and in the sequence $\{x_i\}$),

$$(5.2)\quad \overline{\lim_{\Delta\to 0}}\,\frac{1}{\Delta}\,\overline{\lim_{N}}\,a_N \log E\left[\exp \sum_{m(t_N+t)}^{m(t_N+t+\Delta)} \langle \alpha, a_i b_i(x_i,\xi_i)\, a_N^{-1}\rangle \mid \xi_{m(t_N+t)} = \xi\right] \le \bar{H}_\gamma(x,\alpha,t),$$

where

$$\bar{H}_\gamma(x,\alpha,t) = K^{-1}(t)H_0(x,K(t)\alpha) + \langle \alpha, \bar{b}(x)\rangle + \gamma K^{-1}(t)(|K(t)\alpha| + 1),$$

analogously to the case in Section 4. Define $\bar{L}_\gamma$ to be the dual of $\bar{H}_\gamma$

By (5.2) and the theorem of Gärtner referred to in Theorem 4.2, if $\{x_i\}$ generates $\{\xi_i\}$ and $|x_i - x| \le \delta$, then for Borel A,

$$(5.3)\quad \overline{\lim_{N}}\, a_N \log P\left\{\sum_{m(t_N+t)}^{m(t_N+t+\Delta)} a_i b_i(x_i,\xi_i) \in A \mid \xi_{m(t_N+t)} = \xi\right\} \lesssim -\inf_{\beta\in\bar{A}} \Delta \bar{L}_\gamma(x,\beta/\Delta,t),$$

where the estimate is uniform on any compact (x,ξ) set, as well as in the sequence $\{x_i\}$.

Henceforth $\phi(\cdot)$ is some function in $C_x[0,T]$ with Lipschitz constant $\le K$. Recall the definition of $\{X_n^{N,x}, n \ge N\}$ from (2.9). The sequence generating $\{\xi_i\}$ will be the x_i-arguments in $b_i(x_i,\xi_i)$ in the functions below. It will usually be $\{X_n^{N,x}, n \ge N\}$. Let $\Delta \le \delta/(\gamma + K)$. Define $D^\Delta\phi(t) = \phi(t + \Delta) - \phi(t)$. Then, it follows from (5.3) that (uniformly in each compact t, x, ξ set)

$$(5.4)\qquad \overline{\lim_N}\; a_N \log P\Big\{\Big|\sum_{m(t_N+t)}^{m(t_N+t+\Delta)} a_i b_i(X_i^{N,x},\xi_i) - D^\Delta\phi(t)\Big| \le 2\gamma\Delta \;\Big|\; \xi_{m(t_N+t)} = \xi,$$

$$|X^{N,x}_{m(t_N+t)} - \phi(t)| \le \gamma\Delta\Big\}$$

$$\le - \inf_{\{\beta\,:\,|\beta - D^\Delta\phi(t)| \le 2\gamma\Delta\}} \Delta\bar{L}_\gamma(\phi(t),\beta\,\Delta,t).$$

By [20, Lemma 2.4], (A5.2ii) implies that for given β, we can find β' such that $|\beta - \beta'| \le \gamma\Delta$ and

$$(5.5)\qquad \bar{L}(\phi(t),\beta'\,\Delta,t) \le \bar{L}_\gamma(\phi(t),\beta\,\Delta,t) + \gamma.$$

By using this in (5.4) we can replace the right side by

$$(5.6)\qquad - \inf_{\{\beta\,:\,|\beta - D^\Delta\phi(t)| \le 3\gamma\Delta\}} \Delta\bar{L}(\phi(t),\beta\,\Delta,t) + \gamma\Delta\,.$$

(Since γ can be made as small as desired, the added $\gamma\Delta$ will eventually be dropped.)

Fix $T < \infty$ to be an integral multiple of Δ. Then (uniformly in x, ξ on each compact set) (5.4) and (5.6) yield

$$(5.7)\qquad \overline{\lim_N}\; a_N \log P(d(X^{N,x},\phi) \le \gamma\Delta \mid \xi_N = \xi)$$

$$\le \overline{\lim_N}\; a_N \log P(|X^{N,x}(i\Delta) - \phi(i\Delta)| \le \gamma\Delta,\ i \le T/\Delta \mid \xi_N = \xi)$$

$$\le \sum_0^{T/\Delta - 1} \overline{\lim_N}\; a_N \log P(|D^\Delta(X^{N,x}(i\Delta) - \phi(i\Delta))| \le 2\gamma\Delta \;\Big|$$

$$|X^{N,x}(i\Delta) - \phi(i\Delta)| \le \gamma\Delta,\ \xi_{m(t_N+i\Delta)})$$

$$\le -\Delta \sum_0^{T/\Delta - 1} \inf_{\{\beta\,:\,|\beta - D^\Delta\phi(i\Delta)| \le 3\gamma\Delta\}} \bar{L}(\phi(i\Delta),\beta\,\Delta,i\Delta) + \gamma T$$

Let $\bar{\phi}_\Delta(t) = \phi(i\Delta)$ for $t \in [i\Delta, i\Delta + \Delta)$, and define $A_\Delta^\gamma(\phi) = \{\psi \in C_x[0,T]: \dot{\psi}(t)$ is constant for $t \in [i\Delta, i\Delta + \Delta)$, and $|\dot{\psi}(i\Delta) - D^\Delta\phi(i\Delta)/\Delta| \leq 3\gamma$, $i \leq T/\Delta\}$. Then the r.h.s. of (5.7) may be replaced by

$$(5.8) \qquad -\inf_{\psi \in A_\Delta^\gamma(\phi)} \int_0^T \bar{L}(\bar{\phi}_\Delta(t), \dot{\psi}(t), t)dt + \gamma T.$$

Since $\bar{S}_x(T,\phi) < \infty$ implies that ϕ satisfies a Lipschitz condition that is independent of x, there is $\Delta > 0$ (independent of ϕ and x) such that for ϕ satisfying $\bar{S}_x(T,\phi) < \infty$, we have

$$(5.9) \qquad \begin{aligned} &\sup_{0 \leq t \leq T} |\bar{\phi}_\Delta(t) - \phi(t)| \leq \gamma, \\ &\sup_{\psi \in A_\Delta^\gamma(\phi)} \sup_{0 \leq t \leq T} |\psi(t) - \phi(t)| \leq 4\gamma T. \end{aligned}$$

Using the result of part (c) of Theorem 4.2 we can now complete the estimate just as we did in part (e) of that theorem. Let $s > 0$ and $h > 0$ be given. By picking γ small (which implies $\Delta > 0$ is small) we have that $x \in F$, $\bar{S}_x(T,\phi) \geq s$ implies

$$(5.10) \qquad -\inf_{\psi \in A_\Delta^\gamma(\phi)} \int_0^T \bar{L}(\bar{\phi}_\Delta(t), \dot{\psi}(t), t)dt \leq -\bar{S}_x(T,\phi) + h.$$

Let $\gamma > 0$ now be fixed, and set $\hat{\delta} = \min[\delta, \Delta\gamma]/2$. Define R(x) and choose the $\hat{\delta}$-net of R(x) whose cardinality *is independent of* $x \in F$ *as in* part (c) of the proof of Theorem 4.2. Then using (5.7) through (5.10), for large enough N (and independent of ξ and x) we have

$$
\begin{aligned}
&P_\xi\{d(X^{N,x}, \bar{\Phi}_x(s)) > \delta\} \\
&\quad \leq \sum_1^M P_\xi\{d(X^{N,x}, \phi_i^x) \leq \gamma\Delta\} I_{\{d(\phi_i^x, \bar{\Phi}_x(\cdot)) > \delta\}} \\
&\quad \leq \sum_1^M (\exp(-\bar{S}_x(T, \phi_i^x) + 2h + \gamma T)/a_N)) I_{\{d(\phi_i^x, \bar{\Phi}_x(\cdot)) > \delta\}} \\
&\quad \leq M \exp(-s + 2h + \gamma T)/a_N.
\end{aligned}
$$

(A2.6) follows from this estimate in the same way it followed from (4.16) in Theorem 4.2. □

Example. We present one example of the processes that may be handled by the methods of this subsection. The example includes the adaptive routing process mentioned in the introduction. For additional examples, the reader may refer to [20].

The basic method for proving that an assumption such as (A5.2ii) holds is to first show (2.2b), and to then use (assuming it exists) the continuity properties of the measure induced by $b_i(x, \xi_i)$ as a function of x, which must be uniform when conditioned on ξ_{i-1}. Conditions under which (2.2b) holds are given in Example 4 of Section 2. For more details on that and other examples, the reader is referred to [24].

We next state the assumptions of the example, then discuss them, and then prove (A5.2ii). Assume (i) - (iii) below.

(i) *There are functions* $a_i(x)$, $\rho_i(x,\xi) \geq 0$, $1 \leq i \leq N$, *with* $a_i(x)$ *continuous in* x and $\rho_i(x,\xi)$ *continuous in* x, *uniformly in* ξ. *Furthermore* $\sum_i \rho_i(x,\xi) = 1$ *and each* ρ_i *is either identically zero or bounded from below by some* $c > 0$. *Let*

$$P\{b_n(x,\xi) \in A \mid x,\xi\} = \sum_1^N \rho_i(x,\xi) I_{\{a_i(x) \in A\}}$$

be the measure of the random field $b_n(x,\xi)$.

(ii) *The process* $\{\xi_i^x\}$ *takes values* 1, ..., M, *where* M *does not depend on* x. *Let* P_{ij}^x *and* $P_{i,j}^{x;n}$ *denote the* 1 *and* n*-step transition probabilities of* $\{\xi_n^x\}$.

(iii) *There is* n_0 *(not depending on* x*) such that* P_{ij}^{x,n_0} *is continuous in* x *and strictly positive in* x, i, j.

The third assumption implies that there is a uniform (in x) lower bound on the geometric rate at which the measures induced by ξ_i^x converge to the invariant measure.

Discussion. In many applications, the distribution of the random field $b_n(x,\xi)$ is concentrated on a finite number of points which move continuously with x, and which do not depend on ξ. Then we let these points be the $a_i(x)$, i $\le$ N, and then $\rho_i(x,\xi) = P\{b_n(x,\xi) = a_i(x) \mid x,\xi\}$. In the above cited 'routing example' b_n might take values on 1 (if b_n is an 'indicator' function) or values $a_1(x)$, $a_2(x)$, if it is of the form $a_1(x)I_1 + a_2(x)I_2$, where the I_j are indicators of events which depend on the arrivals, departures, routing realizations, etc., but whose distributions depend on the current values of x, ξ. Both forms have been used in the literature.

Proof of (A5.2ii). By (ii), (iii) above, (2.2b) holds [24]. In order to simplify the notation, we set $n_0 = 1$. The general proof is very similar. Let F be a fixed compact set. Let c > 0 also be a lower bound for P_{ij}^x, $x \in F$.

Given $\gamma > 0$, choose $\delta > 0$ such that $|x - y| \leq \delta$ implies that $|a_i(x) - a_i(y)| \leq \gamma$, $\rho_i(y,\xi) \leq \rho_i(x,\xi)\exp\gamma$ for all ξ and $P^y_{ij} \leq P^x_{ij}\exp\gamma$ for all i, j, and $x \in F$.

For fixed x, let $\{x_i\}$ satisfy the hypothesis of (A5.2ii). The transition probability used to get the ξ_n in $b_n(x_n,\xi_n)$ will be $P^{x_n}_{\xi_{n-1},j}$. Thus $\{x_i\}$ (or whatever sequence replaces it below) generates $\{\xi_n\}$. Then for $\xi = \xi_0$,

$$E_\xi \exp\langle\alpha, \sum_1^n b_i(x_i,\xi_i)\rangle$$

$$(*) \qquad = E_\xi \exp\langle\alpha, \sum_{i=1}^{n-1} b_i(x_i,\xi_i)\rangle E_\xi[\exp\langle\alpha, b_n(x_n,\xi_n)\rangle \mid \xi_{n-1},x_n]$$

$$= E_\xi\left[\exp\langle\alpha, \sum_{i=1}^{n-1} b_i(x_i,\xi_i)\rangle\right]\left[\sum_{j=1}^{N}\sum_{k=1}^{M} \exp\langle\alpha,a_j(x_n)\rangle\rho_j(x_n,k)P^{x_n}_{\xi_{n-1},k}\right].$$

Now, for all x_n, ξ_{n-1}, the last bracketed term is bounded above by

$$(**) \qquad \sum_{j=1}^{N}\sum_{k=1}^{M} \exp\langle\alpha,a_j(x)\rangle\rho_j(x,k)P^x_{\xi_{n-1},k}\exp(|\alpha| + 2)\gamma.$$

Using (**) in (*) and continuing to iterate backwards to approximate all the x_i by x plus an 'error' yields the upper bound to (*) of

$$E_\xi\exp\left[\langle\alpha, \sum_1^n b_i(x,\xi_i^x)\rangle + n(|\alpha| + 2)\gamma\right].$$

(A5.2ii) follows from this and the convergence in (2.2b).

5.3. The Constrained Algorithm

We now outline the extension of Theorems 3.1 and 4.2 for the algorithm (1.2), where G is a convex set which is the closure of its interior and its boundary consists of a finite number of smooth (C^2) sections. References [18], [19] contain a detailed discussion of large deviations bounds (upper and lower) for constrained algorithms, under constant gains $a_n \equiv a > 0$. (Reference [18] is an abbreviated form of [19].) The technique used there is an adaptation of the method of Freidlin [12] for the unconstrained case. In order to simplify the development, we will use the assumptions of [18], [19], and discuss the main questions concerning the adaptation of the proof there to the present case. First, we define the mean projected dynamics for

$$X_{n+1} = \pi_G(X_n + b(X_n,\xi_n)).$$

The processes $\{X_n^{N,x}\}$, $\{X_n^{\psi,N}\}$ and the various linear interpolations are all defined as they were in (2.10), (2.11), (4.7) and (4.8). All neighborhoods and sets used below are relative to G.

For $x \in G$ and $v \in R^r$, define the 'projection' of v at x

$$\pi_G(x,v) = \lim_{\Delta \to 0} [\pi_G(x + \Delta v) - x]/\Delta.$$

Define the set of outer normals to ∂G at x:

$$n(x) = \{\gamma: \text{for all } y \in G,\ \langle \gamma, x-y \rangle \geq 0,\ |\gamma| = 1\}.$$

Note that [26, Lemma 4.6] $\pi_G(x,v)$ equals v if $x \in G^0$ (the interior of G) or $x \in \partial G$ and $\sup_{\gamma \in n(x)} \langle \gamma, v \rangle < 0$ (i.e., where v points inward). In general, it equals

$v - \langle v,\gamma^* \rangle \gamma^*$ if $x \in \partial G$, $\sup_{\gamma \in n(x)} \langle \gamma, v \rangle \geq 0$ and γ^* is the (a) maximizer.

Define $\bar{H}(x,\alpha,t)$ and $\bar{L}(x,\beta,t)$ by (2.7) and define the 'constrained' L-functional by

$$\bar{L}_G(x,\beta,t) = \inf_{v:\pi_G(x,v)=\beta} \bar{L}(x,v,t).$$

For $x \notin G$ or if the infimizing set is empty, set $\bar{L}_G(x,\beta,t) = +\infty$. Then $\bar{L}_G(x,\beta,t) = \bar{L}(x,\beta,t)$ if $x \in G^0$ or if $x \in \partial G$ and $\langle \gamma,\beta \rangle < 0$ for all $\gamma \in n(x)$ (i.e., β points to the interior of G). If $x \in \partial G$ and there is $\gamma \in n(x)$ such that $\langle \gamma,\beta \rangle > 0$ (β points 'out' of G), then $\bar{L}_G(x,\beta,t) = \infty$. The interesting case is when $\sup_{\gamma \in n(x)} \langle \gamma,\beta \rangle = 0$; i.e., β points 'along the boundary.' In this case, there is a true (nontrivial) minimization. Since $\bar{L}(x,\beta,t)$ is l.s.c. in β and $\bar{L}(x,\beta,t) \to \infty$ as $|\beta| \to \infty$ (under the assumptions to be used), the infima is attained. Define

$$\bar{S}^*_{G,x}(T,\phi) = \int_0^T \bar{L}_G(\phi(s),\dot{\phi}(s),s)ds,$$

and the ODE for the projected mean dynamics

$$\dot{x} = \pi_G(x,\bar{b}(x)), \tag{5.11}$$

where $\bar{b}(\cdot)$ is defined as in (A2.2).

One of the main difficulties as well as points of interest for the constrained algorithm is that in many applications the escape of $\{X_n\}$ from a neighborhood of a stable point of (5.11) will be essentially along the boundary, and when such neighborhoods are entered from the outside it is often essentially along the boundary as well.

We will use the assumption

A5.3. (i) *The noise* $\{\xi_n\}$ *is exogenous.*

(ii) $b(x,\xi)$ *is bounded (by* K) *and is Lipschitz continuous in* x, *uniformly in* ξ.

(iii) *Partition* $b(x,\xi)$ *into two parts, of dimension* s *and* r − s, *resp. Define*

$$\frac{1}{n}\operatorname{cov}\sum_{j=1}^{n}[b(x,\xi_{N+j}) - \bar{b}(x)] = \Sigma^{N,n}(x) = \begin{bmatrix} \Sigma_{11}^{N,n}(x) & \Sigma_{12}^{N,n}(x) \\ \Sigma_{21}^{N,n}(x) & \Sigma_{22}^{N,n}(x) \end{bmatrix}.$$

Then either (a) *(the non-degenerate case)* $\underline{\lim}_{N,n}\,\Sigma^{N,n}(x) > 0$ *(in the sense of positive definite matrices) or (the degenerate case)*

$$\text{(b)}\quad \lim_{N,n}\Sigma_{11}^{N,n}(x) = \lim_{N,n}\Sigma_{12}^{N,n}(x) = \lim_{N,n}\Sigma_{21}^{N,n}(x) = 0 \quad \textit{and} \quad \underline{\lim}_{N,n}\Sigma_{22}^{N,n}(x) > 0.$$

Remark. A5.3(iii) is not particularly restrictive in applications, since many algorithms divide naturally into components which are not directly affected by noise and those which are in a 'non-degenerate' manner. It and A5.3(ii) were used in [18,19] to prove the l.s.c. of $S_{G,x}(T,\phi)$, the action functional for the constant gain $a_n \equiv a > 0$ case [19, Theorem 2]. In [18,19], we required the set $U(x) = \{\beta: L(x,\beta) < \infty\}$ (or its analog in the degenerate case A5.3(iiib)) to be continuous in the Hausdorff topology, but, in fact, this follows from A5.3(ii). The analog of this 'Hausdorff continuity' condition for our case will always hold by A5.3(ii) wherever it is needed to adapting the proofs in [18,19] to our $a_n \to 0$ case.

Theorem 5.3. *Let* (5.8) *have a unique solution for each initial condition in* G. *Let* θ *be an asymptotically stable point of* (5.8) *with domain of attraction* $\Lambda \subset G$, *and let* $\{X_n\}$ *enter (infinitely often* w.p.1) *a compact set* $D(\theta) \subset \Lambda$. *Assume* (A2.1), (A2.3), (A2.5), (A2.7) *and* (A5.3). *Then* $X_n \to \theta$ w.p.1.

If we assume in addition that given $\epsilon > 0$ *there is* $\bar{N} < \infty$ *such that* $a_i/a_N < 1 + \epsilon$ *for all* $i \geq N \geq \bar{N}$ *, then*

$$(5.12)\qquad \overline{\lim_N}\, a_N \log P\{X_n \notin D(\theta),\ \textit{some}\ n \geq N \,|\, |X_N - \theta| \leq \delta\} \leq - \inf_{\substack{\phi:|\phi(0)-\theta| \leq \delta \\ \phi(t) \notin D(\theta) \text{ for some } t<\infty}} \bar{S}_{G,\phi(0)}(t,\phi).$$

Remark. Rate of convergence results for the constrained algorithms are not available via the classical stochastic approximation method of 'local linearization'. This makes estimates of the form (5.12) particularly important.

Remarks on the Proof. The argument closely follows the lines of the argument of Sections 3 and 4. In [19, Theorem 2], for the 'constant gain' case, the l.s.c. of $\bar{S}_{G,x}(T,\phi)$ was proved. Purely notational changes in the proof there gives the l.s.c. in ϕ of $\bar{S}_{G,\phi(0)}(T,\phi)$. Since $\pi_G(x,v) = \beta$ implies $|v| \geq |\beta|$, the compactness of (for compact F)

$$\bigcup_{x\in F} \{\phi: \bar{S}_{G,x}(T,\phi) \leq s\}$$

is proved as it was in Lemma 3.1(ii). Note also that $\bar{S}_{G,x}(T,\phi) = 0$ iff $\bar{L}_G(\phi(s),\dot{\phi}(s),s) = 0$ a.s. By the definition of $\bar{L}_G$ and the fact that $\bar{L}(x,\beta,s) = 0$ iff $\beta = \bar{b}(x)$, $\bar{L}_G(x,\beta,s) = 0$ iff $\beta = \pi_G(x,\bar{b}(x))$. Therefore $\bar{S}_{G,x}(T,\phi) = 0$ iff $\dot{\phi}(s) = \pi_G(\phi(s),\bar{b}(\phi(s)))$ a.s. These remarks give us the 'constrained case' analog of Lemma 3.1. Then, if (A2.6) held, with $\bar{S}_{G,x}$ replacing $\bar{S}_x$, we would have our convergence theorem.

Under the smoothness condition in (A5.3ii), the proof in [18,19] can be adapted to get the necessary form of (A2.6), in much the same way that Freidlin's proof in [12] was adapted to get the proof of Theorem 4.2.

6 APPENDIX 1. ON THE DIFFERENTIABILITY OF $H(x,\alpha)$ at $\alpha = 0$ in (A2.3)

When the state process $\{X_n\}$ is Markov, or when it is one component of a Markov process (such as our 'state-dependent noise' process $\{X_i, \xi_i\}$) satisfying certain 'uniformly recurrent' conditions, it is possible to prove the differentiability of $H(x,\alpha)$ for a wide class of such processes by using analytical techniques and the characterization of $H(x,\alpha)$ as the log of the eigenvalue of largest modulus of an operator associated to the process (as in [24]). See [24] for details on how this approach may be used in a general setting. However, for many of the processes arising in the study of stochastic systems the assumptions required by this approach do not hold. As a very common example, one may consider the ARMA model to be discussed below.

In this section, we outline a method for proving the differentiability of $H(x,\alpha)$ at $\alpha = 0$ that is based on well known 'level 2' and 'level 3' large deviations results and which is general enough to cover many of the non-Markov processes encountered in recursive algorithms.

In applications, we would not want to be concerned with the abstract level of results in this section. But, they make it clear that the α-differentiability assumption is not restrictive and can be treated in many different ways. We work only with the exogenous noise case and stationary and continuous deterministic random fields for simplicity of exposition. Define the sample occupation measure (over the Borel sets Γ)

$$L_N(\Gamma,\omega) = \frac{1}{N}\sum_{1}^{N} I_{\{b(x,\xi_i)\in\Gamma\}}.$$

and the space $\mathcal{M}$ of probability measures on R^r endowed with the topology of weak convergence. The $L_N(\cdot,\omega)$ are in $\mathcal{M}$. Assume the following large deviations estimate (x is fixed throughout):

A6.1. *There is a l.s.c. non-negative functional I_x on $\mathcal{M}$ such that the sets $\{\nu \in \mathcal{M}: I_x(\nu) \le s\}$ are compact for $s < \infty$, and for Borel $A \subset \mathcal{M}$ and each ω (w.p.1):*

$$\overline{\lim_N}\ \frac{1}{N}\log P_{\mathcal{F}}\{L_N \in A\} \le -\inf_{\nu\in\bar{A}} I_x(\nu)$$

Sufficient conditions for (A6.1) are contained in many places [22], [23]. Assume

A6.2. *There is a unique measure $\bar{\nu}_x \in \mathcal{M}$ such that $I_x(\bar{\nu}_x) = 0$.*

It follows from (A6.1) and (A6.2) that $L_N(\cdot,\omega)$ converges (w.p.1) to $\bar{\nu}_x(\cdot)$.

We now show that (A6.1) and (A6.2) imply the desired α-differentiability. Then an example will be given, and the approach discussed.

By Varadhan's theorem on the asymptotic evaluation of integrals [21] and the boundedness and continuity of $b(x,\cdot)$, the following inequality holds (w.p.1):

$$(6.1)\qquad \overline{\lim_N}\ \frac{1}{N}\log E_{\mathcal{F}_0}\exp\langle\alpha,\sum_1^N b(x,\xi_i)\rangle \le \sup_\nu\left[\int\langle\alpha,y\rangle\nu(dy) - I_x(\nu)\right] \equiv H^*(x,\alpha).$$

Obviously $H(x,\alpha) \le H^*(x,\alpha)$. Since both functions are convex and $H(x,0) = H^*(x,0) = 0$, $H(x,\alpha)$ is α-differentiable at $\alpha = 0$ if $H^*(x,\alpha)$ is

Next note that

$$(6.2)\qquad H^*(x,\alpha) = \sup_{\beta} \sup_{\{\nu \in \mathcal{M}:\ \int y\nu(dy)=\beta\}} \left[\int \langle \alpha, y\rangle \nu(dy) - I_x(\nu)\right]$$

$$= \sup_{\beta}\,[\langle \alpha,\beta\rangle - L^*(x,\beta)],$$

where

$$L^*(x,\beta) = \inf_{\{\nu \in \mathcal{M}:\ \int y\nu(dy)=\beta\}} I_x(\nu).$$

Since $H^*(x,0) = 0$, β^* is a subdifferential of $H^*(x,\alpha)$ at $\alpha = 0$ if

$$(6.3)\qquad H^*(x,\alpha) - \langle \alpha,\beta^*\rangle \geq 0, \text{ all } \alpha.$$

But (6.3) holds iff $L^*(x,\beta^*) = 0$ since H^* is the Legendre transform of L^*. Since $H^*(x,\ )$ is convex, it is differentiable at $\alpha = 0$ iff the set of subdifferentials at $\alpha = 0$ contains only one element. By (A6.2), $\beta^* = \int y\bar{\nu}_x(dy)$ is the unique value of β for which $L^*(x,\beta) = 0$. Thus $\beta^* = \int y\bar{\nu}_x(dy)$ is the unique subdifferential, and the α-differentiability is proved. Note that $\beta^* = \bar{b}(x)$, as defined in (A2.2).

<u>Discussion</u>. We have phrased our requirement in terms of $H(x,\alpha)$ at $\alpha = 0$, but as shown above this is obviously *equivalent to the uniqueness of the* β^* *satisfying* $L(x,\beta^*) = 0$. The reason for our choice is that in most of the work on large deviations for dynamical systems [12], as well as the work generalizing Cramer's original paper [16], [24], [25], the differentiability of $H(x,\alpha)$ in α is taken as a fundamental assumption. As a consequence, this was the condition that was typically verified for a given noise process. See for example Lemma 3.4 of [24].

We illustrate the method with an example.

Example. Suppose that $b(x,\cdot)$ is continuous, and that $\{\xi_n\}$ is a stationary ARMA process with representation

$$A_0\xi_n + A_1\xi_{n-1} + \cdots + A_{d_1}\xi_{n-d_1} = B_0\psi_n + B_1\psi_{n-1} + \cdots + B_{d_2}\psi_{n-d_2}, \tag{6.4}$$

where $\{\psi_i\}$ is a sequence of zero mean, bounded, i.i.d. random variables. For simplicity, we assume both ξ_i and ψ_i take values in R^r. It is also assumed that the roots of $\det(A_0 + A_1 s + \cdots + A_{d_1} s^{d_1})$ lie outside of the closed unit disc.

Define $S = (R^r)^{\mathbb{Z}}$ (the space of infinite sequences with values in R^r), and consider the mapping $F: S \to S$ defined by ($(F(\ ))_j$ denotes the jth component)

$$F(\{s_i\})_j = b(x,p_j)$$

where $\{s_i\}$ and $\{p_i\}$ are related by

$$A_0p_n + \cdots + A_{d_1}p_{n-d_1} = B_0s_n + \cdots + B_{d_2}s_{n-d_2}.$$

We can metrize S in such a way that F is continuous (and in fact uniformly continuous on a subset $A \subset S$ such that $\{\psi_i\} \subset A$ w.p.1). It is then relatively straightforward to show that (A6.1) and (A6.2) follow from the (so-called 'level 3') large deviations results for the process $\{\psi_i\}$ that are given in [23], under a suitable application of the 'contraction principle' (a 'continuous mapping' technique) [21; Section 2]. We omit all details here, since they would take us too far afield, and the techniques are known in large deviations theory.

In general, if a given process $\{\xi_i\}$ can be represented as a continuous transformation of a simpler process $\{\psi_i\}$ for which the appropriate 'level 3' results exist, then we may obtain (A6.1) and (A6.2) via the 'contraction principle'. In the course of doing so, we also verify A2.1, with $H(x,\alpha)$ there replaced by $H^*(x,\alpha)$ of (6.1).

Although this approach may seem abstract, it in fact rather easily yields the α-differentiability for a wide variety of the noise processes of interest in stochastic systems theory, which often *do* have such a representation.

7 APPENDIX 2. PROOF OF THEOREM 5.1 FOR $b_i(x,\xi) = b_i(x)$, SCALAR CASE

We adapt the proof in [17] for the constant $a_n \equiv a > 0$ case. The details for the full Theorem 5.1 use a similar adaptation. Define $G_x(a) = P\{b(x, \xi) \le a\}$, and $G_x^{-1}: [0,1] \to R$ by $G_x^{-1}(v) = \sup\{a : G_x(a) \le v\}$. Let $\{v_n\}$ and $\{\rho_n\}$ be mutually independent sequences of random variables, each i.i.d., with the v_n uniformly distributed on $[0,1]$, and the ρ_n Gaussian with mean zero and variance $\sigma^2 > 0$. Let $p_\sigma(\cdot)$ denote the density of ρ_n, and $G_{\sigma,x}(\cdot)$ the convolution of $G_x(\cdot)$ and the distribution function of ρ_n. Then

$$\frac{d}{da} G_{\sigma,x}(a) = \int p_\sigma(a-b)\, dG_x(b)$$

is uniformly positive on each bounded (x,a) set.

Define $G_{\sigma,x}^{-1}(\cdot)$ as $G_x^{-1}(\cdot)$ was but using $G_{\sigma,x}(\cdot)$. Let F_1 denote a compact set. It follows from the weak continuity in (A5.1) that $G_{\sigma,x}^{-1}(v)$ is continuous in $(x,v) \in R \times (0,1)$ and that given $\Delta \in (0,1/2)$, $G_{\sigma,x}^{-1}(v)$ is uniformly continuous on $F_1 \times [\Delta, 1-\Delta]$. Define $F_n(x) = G_x^{-1}(v_n)$, $F_{\sigma,n}(x) = G_{\sigma,x}^{-1}(v_n)$. Now, analogous to what was done in Section 4, for $x \in R^r$ and $n \ge N$, define the auxiliary processes in (7.1) (here also $\psi(\cdot)$ is piecewise constant on $[0,T]$ with intervals of constancy $[i\Delta, i\Delta + \Delta)$ and we write $\psi_n^N \equiv \psi(t_n - t_N)$, $n \ge N$):

$$\text{(7.1a)} \qquad X_{n+1}^{N,x} = X_n^{N,x} + a_n F_n(X_n^{N,x}), \qquad X_N^{N,x} = x,$$

$$\text{(7.1b)} \qquad \tilde{X}_{n+1}^{N,x} = \tilde{X}_n^{N,x} + a_n F_n(X_n^{N,x}) + a_n \rho_n, \qquad \tilde{X}_N^{N,x} = x,$$

$$\text{(7.1c)} \qquad X_{\sigma,n+1}^{N,x} = X_{\sigma,n}^{N,x} + a_n F_{\sigma,n}(X_n^{N,x}), \qquad X_{\sigma,N}^{N,x} = x,$$

$$\text{(7.1d)} \qquad X_{\sigma,n+1}^{\psi,N} = X_{\sigma,n}^{\psi,N} + a_n F_{\sigma,n}(\psi_n^N), \qquad X_{\sigma,N}^{\psi,N} = x, \ n \ge N$$

Owing to the way that G_x^{-1} was constructed, the distributions of the process defined by (7.1a) are equal to those of the $\{X_n^{N,x}, n \geq N\}$ process defined in (2.9). (In fact, the discussion above indicates how the random fields $b_i(x)$ could be constructed.)

By the definitions of $G_x^{-1}(\cdot)$ and $G_{\sigma,x}^{-1}(\cdot)$, the distributions of $\{\tilde{X}_n^{N,x}, n \geq N\}$ and $\{X_{\sigma,n}^{N,x}, n \geq N\}$ are the same, and we will work with the latter. Note that the $\{\rho_n\}$ no longer appears in $\{X_{\sigma,n}^{N,x}, n \geq N\}$. For $\delta > 0$ and large N,

$$(7.2) \qquad P\left\{\sup_{N \leq n \leq m(t_n+T)} |\tilde{X}_n^{N,x} - X_n^{N,x}| \geq \delta\right\} = 2P\left\{\sup_{N \leq n \leq m(t_n+T)} \sum_N^n a_i\rho_i \geq \delta\right\}$$

$$\leq \exp - \delta/2a_N K_1 T\sigma^2,$$

where K_1 is an upper bound for $\sup_{N \leq n \leq m(t_N+T)} a_n/a_N$ for large N.

The equivalence (in distribution) of the processes defined by (7.1b) and (7.1c) and (7.2) essentially allows us to prove the theorem by using a large deviations upper bound for (7.1c) -- which is 'smoothed', since $G_{\sigma,x}^{-1}(v)$ is x-continuous. A large deviations upper bound of the type obtained in Section 4 can readily be obtained for (7.1c), via the intermediary process (7.1d) (as in Section 4). Henceforth x is confined to a compact set F_1.

Next, let $X^{N,x}(\cdot)$, $X_\sigma^{N,x}(\cdot)$ and $X_\sigma^{\psi,N}(\cdot)$ denote the piecewise linear interpolations as in (2.11), but for the processes defined by (7.1a,c,d). Recall part (d) of Theorem 4.2. The following set inclusion (4.12) was the key part of the proof

$$(4.12) \qquad \{d(X^{N,x},\phi) \leq \delta\} \subset \{d(X^{\psi,N},\phi) \leq \delta_2\}.$$

Here, we work with the set inclusion (7.3) instead (see below for proof) for appropriate δ and δ_2.

$$(7.3) \qquad \{d(X^{N,x},\phi) \le \delta\} \subset \{d(X_\sigma^{\psi,N},\phi) \le \delta_2\} \cup N,$$

where $P\{N\} \le \exp - M_\sigma/a_N$ with $M_\sigma \to \infty$ as $\sigma \to 0$ and not depending on ϕ or x. (For the general ξ-dependent b_i, we use the conditional probability, as in Theorem 4.2, and all upper bounds are uniform in ω w.p.1.)

Define

$$\bar{H}_\sigma(x,\alpha,t) = \bar{H}(x,\alpha,t) + K(t)\alpha^2\sigma^2/2$$

and the associated L and S_x functionals $\bar{L}_\sigma$ and $\bar{S}_{\sigma,x}$. Owing to the added ρ_n in (7.1b), $\bar{H}_\sigma$ is the proper H-functional for $\hat{X}^{N,x}(\cdot)$ and for $X_\sigma^{N,x}(\cdot)$. It is enough to work with the inclusion (7.3) instead of (4.12) as in Theorem 4.2, owing to the inequality (7.2) and the equivalence (in distribution) of the processes $\hat{X}^{N,x}(\cdot)$ and $X_\sigma^{N,x}(\cdot)$. Now, the same arguments that were used in Theorem 4.2 now imply Assumption (A2.6), but with $\bar{S}_x$ replaced by $\bar{S}_{\sigma,x}$. By [17, Lemma 1],

$$\underline{\lim}_{\sigma\to 0} \inf_{\phi\in\bar{A}} \bar{S}_{\sigma,x}(T,\phi) \ge \inf_{\phi\in\bar{A}} \bar{S}_x(T,\phi).$$

The last two sentences yield the theorem. Thus, only the set inclusion (7.3) needs to be shown. This inclusion is proved in exactly the same way as (2.6) in [17] is proved, with a_j or a_N replacing ϵ, $X_\sigma^{N,x}$ replacing X_σ^ϵ and $X_\sigma^{\psi,N}$ replacing $X_\sigma^{\epsilon,\psi}$, and we omit the details. □

REFERENCES

[1] M. B. Nevelson and R. Z. Hasminskii, Stochastic Approximation and Recursive Estimation, Trans. of Math. Monographs, V. 47, AMS, 1972.

[2] M. T. Wasan, Stochastic Approximation, Cambridge Univ. Press, 1969.

[3] H. J. Kushner, D. S. Clark, Stochastic Approximation Methods for Constrained and Unconstrained Systems, 1978, Springer-Verlag, Berlin.

[4] M. Metivier and P. Priouret, Applications of a Kushner and Clark lemma to general classes of stochastic algorithms, IEEE Trans. on Inf. Theory, IT-30, (1984), pp. 140-151.

[5] H. J. Kushner, Approximation and Weak Convergence Methods for Random Processes with Applications to Stochastic Systems Theory, MIT Press, Cambridge, Mass. (1984).

[6] H. J. Kushner and H. Huang, Rates of convergence for stochastic approximation type algorithms, SIAM J. on Control and Opt., 19 (1981), pp. 87-105.

[7] L. Ljurg, Analysis of recursive stochastic algorithms, IEEE Trans. on Automatic Control, AC-22, 1977, 551-575.

[8] H. J. Kushner, An averaging method for stochastic approximations with discontinuous dynamics, constraints, and state dependent noise, in Recent Advances in Statistics (Ed. Rizvi, Rustagi and Siegmund), Academic Press, 1983.

[9] A. P. Koroselev, Stochastic Recurrent Processes, Nauka, Moscow, 1984. (In Russian.)

[10] H. J. Kushner, Asymptotic behavior of stochastic approximation and large deviations, LCDS Report #83-1, Brown Univ., IEEE Trans. Automat. Control AC-29 (1984), pp. 984-990.

[11] P. Dupuis, H. J. Kushner, Stochastic approximations via large deviations: asymptotic properties, SIAM J. on Control and Opt., 23 (1985), 675-696.

[12] M. I. Freidlin, The averaging principle and theorems on large deviations, Russian Math. Surveys, 33:5 (1978), 117-176.

[13] H. J. Kushner, Robustness and approximation of escape times and large deviations estimates for systems with small noise effects, LCDS Report #82-5, Brown Univ., SIAM J. Appl. Math., 44 (1984), pp. 160-162.

[14] A. D. Ioffe and V. M. Tikhomirov, Theory of Extremal Problems, North-Holland, New York, 1979.

[15] M. I. Freidlin, and A. D. Wentzell, Random Perturbations of Dynamical Systems, Springer-Verlag, Berlin, 1984.

[16] J. Gärtner, On large deviations from the invariant measure, Theory of Prob. and its Applications, 22 (1977), 24-39.

[17] P. Dupuis and H. J. Kushner, Upper bounds for large deviations for non-smooth stochastic difference equations, submitted to The Annals of Probability.

[18] H. J. Kushner and P. Dupuis, Constrained stochastic approximation via the theory of large deviations, Adaptive Statistical Procedures and Related Topics, (Ed. Van Ryzin), IMS Lecture Notes - Monograph Series, Vol. 8, 1986.

[19] P. Dupuis and H. J. Kushner, Asymptotic behavior of constrained stochastic approximations via the theory of large deviations, Probability Theory and Related Fields, to appear.

[20] P. Dupuis, Large deviations analysis of some recursive algorithms with state-dependent noise, Ann. of Probability (submitted)

[21] S. R. S. Varadhan, Large Deviations and Applications, CBMS Regional Series in Applied Mathematics, 46, SIAM, Philadelphia, 1984.

[22] S. Orey and S. Pelikan, Large deviations for stationary processes, IMA Preprint Series #227, Institute for Mathematics and its Applications, University of Minnesota, 1986.

[23] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time, Comm. Pure and Appl. Math.; Pt. I, 28, (1975), pp. 1-47; Pt. II, 28 (1975), pp. 279-301; Pt. III, 29 (1977), pp. 389-461; Pt. IV, 36 (1983), pp. 183-212.

[24] I. Iscoe, P. Ney, E. Nummelin, Large deviations of uniformly recurrent Markov additive processes, Adv. in Applied Math. 6 (1985) 373-412.

[25] H. Cramér, Sur un nouveaux theorème-limite de la theorie des probabilités, in Colloque consacré à la théorie des probabilités, Vol. 3, Hermann, Paris, 1938.

[26] P. Dupuis, Large deviations analysis of reflected diffusions and constrained stochastic approximation algorithms in convex sets, to appear in Stochastics.

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